Proceedings of the 4th International Conference on Computing Innovation and Applied Physics
DOI: 10.54254/2753-8818/83/2025.20240

Historical and Future Development of Lie Group

Chengran Su[1,a,*]

[1]*Beijing NO.101 Middle School, 11 Yiheyuanlu, Haidian District, Beijing, China*
a. boris.su25@beijing101id.com
**corresponding author*

Abstract: This article presents a concise historical overview of Lie groups, underscoring their pivotal role in mathematics. It traces the development from Augustin-Louis Cauchy to Hermann Weyl, illustrates the modern definition and integrated features of Lie groups, and highlights Lie groups as essential bridges among mathematical disciplines. This article also introduces applications of Lie groups in mathematics. Lie groups have wide applications in geometry, differential equations, representation theory, and beyond, making them a unifying concept. Lie groups are also very diverse in the future developments, and can be applied in geometry, algebra, and analysis in mathematics, as well as in deep learning in computers, and in nuclear and particle physics.

Keywords: Lie Groups, Historical Development, Sophus Lie.

1. Introduction

Lie groups have a long and important history, with far-reaching developments in mathematics and physics. Sophus Lie revolutionized the study of continuous transformation groups by introducing the concepts of Lie groups and Lie algebras in his seminal work Theorie der Transformationsgruppe in the late 19th century. Subsequent contributions by mathematicians such as Elie Cartan, Claude Chevalley, and Armand Borel further advanced the theory. Lie groups play a fundamental role in a number of fields, including algebraic geometry, number theory, quantum mechanics, particle physics and general relativity. They provide a powerful mathematical framework for understanding symmetry and have a wide range of applications in modern mathematics and physics. This article will give an account of the important discoveries in the development of Lie groups over the past centuries, and describe the far-reaching impact of this discipline on the development of mathematics and physics.

2. Important Historical Development of Lie Groups

2.1. Sophus Lie and Theorie der Transformationsgruppe. Lie groups

First appeared in 1874 and were first mentioned by the Norwegian mathematician Sophus Lie in his book Theorie der Transformationsgruppe [5]. In the English translation of Theorie der Transformationsgruppe, as modernized by Joel Merker [6], in Chapter 3, ”Fundamental Differential Equations for Finite Continuous Transformation Groups,” the original concept of Lie groups is introduced.

© 2025 The Authors. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (https://creativecommons.org/licenses/by/4.0/).

DOI: 10.54254/2753-8818/83/2025.20240

Lie groups were defined as finite continuous local transformation groups, represented by a family of local analytic diffeomorphisms:

$$x\dot{i} = fi\ (x;a),\ i= 1, \ldots ,n,$$

parametrized by a finite number r of parameters a1 , . . . , ar , satisfying closure under composition and inversion:

$$fi\ (f(x;a);b) = fi\ (x;m(a,b)) \text{ and } xi = fi\ (x' \ ;i(a)).$$

This definition provided the basis for understanding how Lie groups represent continuous symmetries in mathematics and physics.

2.2. Modern Definition of Lie Groups. The modern definition of Lie

groups is primarily attributed to Hermann Weyl in his book The Classical Groups: Their Invariants and Representations, published in 1928 [1]. Weyl built on decades of earlier work, refining the concept of Lie groups into a rigorous and abstract definition.

A Lie group is defined as a differentiable manifold G which is also a group, where the multiplication map G × G ⟶ G and the inverse map G ⟶ G are differentiable. Formally:

$$m : G \times G \qquad \text{(multiplication)}$$

G is a Lie group if it satisfies: $\{i : G \to G \longrightarrow \mathrm{G} \quad \text{(inverse)}$

where both m and i are smooth maps.

This modern definition underpins much of the contemporary understanding of Lie groups, linking group theory with differential geometry and analysis.

2.3. Integrated features of Lie groups

Table 1 shows that the evolution of the concept of Lie groups over the centuries is a remarkable synthesis of analysis, geometry, and algebra, showcasing the interconnectedness of these mathematical disciplines. The evaluation begins in the 19th century with the contributions of AugustinLouis Cauchy and Sophus Lie, who laid the foundation for Lie groups, integral to the mathematical discipline as a whole. Cauchy's work in 1821, Cours d'Analyse de l'École Royale Polytechnique[2], marked a significant starting point by introducing the theory of continuous groups. This concept served as the bedrock for the future development of Lie groups, merging analysis and geometry.

The evolution continued in the 19th century with the pivotal role played by Felix Klein. Klein's contributions to Lie groups and their applications in geometry, as documented in Vorlesungen ¨uber Nicht-Euklidische Geometrie [3], highlighted the growing importance of Lie groups as a bridge between algebra, geometry, and analysis. These groups increasingly gained recognition as powerful tools for understanding the connections between various mathematical structures.

Table 1: Definitions of Lie Groups

Date	Author	Contribution	Literature
1842	Augustin-Louis Cauchy	Cauchy made significant contributions to the theory of continuous groups, which laid the foundation for Lie groups.	Cours d'Analyse de l'École Royale Poly-technique

DOI: 10.54254/2753-8818/83/2025.20240

1870	Sophus Lie	Sophus Lie introduced the concept of continuous transformation groups, which are now known as Lie groups.	Various works, including Theorie der Transformationsgruppen
1872	Felix Klein	Felix Klein contributed to the development of Lie groups and their applications in geometry.	Euklidische Geometrie
1928	Hermann Weyl	Hermann Weyl provided a more modern and abstract definition of Lie groups using matrix groups and algebraic structures.	Gruppentheorie and Quantenmechanik
1951	Andr´e Weil	Andr´e Weil made substantial contributions to the theory of Lie groups and their representations.	L'int´egration dans les groupes topologiques et ses applications(second addition)

During the 20th century, there was a transformation in the understanding of Lie groups, with Hermann Weyl's work being particularly influential. Weyl provided a more modern and abstract definition of Lie groups by employing matrix groups and algebraic structures [1], effectively merging algebra and geometry. This redefinition broadened the scope of Lie groups, illustrating their ability to unify different branches of mathematics.

The evolution of Lie groups culminated with the contributions of Andr´e Weil in 1940. Weil made substantial strides in the theory of Lie groups and their representations, underscoring the crucial role of these groups in mathematical analysis. His work, L'int´egration dans les groupes topologiques [4], demonstrated how Lie groups had become central to the study of various mathematical structures, acting as a bridge between different mathematical disciplines.

In summary, the evolution of Lie groups over the centuries underscores their vital position in the mathematical discipline, acting as a unifying force that connects analysis, geometry, and algebra. Extending from the original concept of differentiable groups, Lie groups have become a great mathematical concept linking the larger mathematical disciplines of the present day, as well as supporting the problems of the Langlands Program of the past few years. Their comprehensive character in mathematics has undoubtedly made them a great link between the mathematical disciplines as a whole.

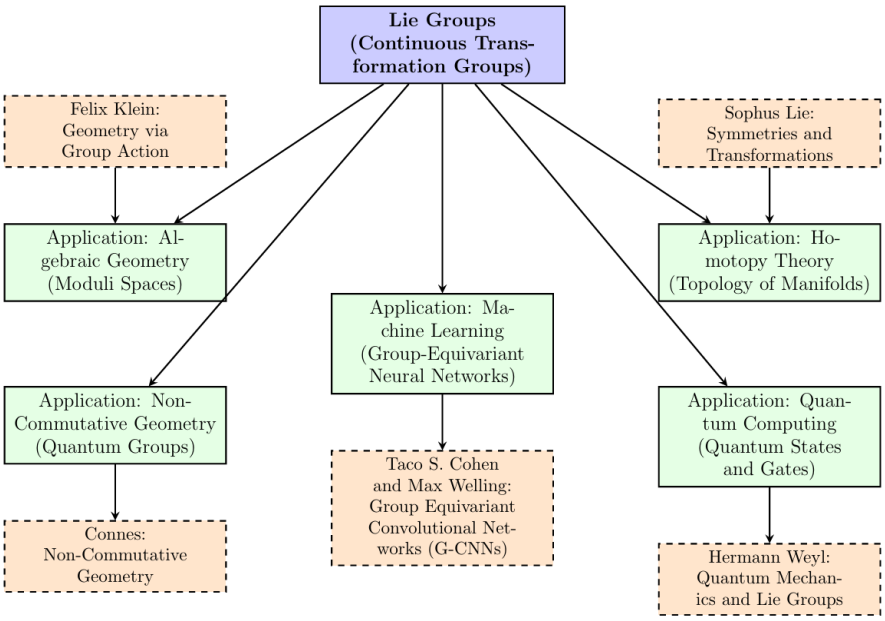

Figure 1: Lie Groups and Their Applications with Theoretical Foundations

DOI: 10.54254/2753-8818/83/2025.20240

3. Applications of Lie Groups

3.1. Applications of Lie Groups in Mathematics

Lie groups are smooth manifolds that form topological groups under multiplication and inverse operations. They provide a natural setting for studying symmetry and differential equations, and have been extensively used in various branches of mathematics such as geometry, analysis, and representation theory. The main goal of Lie groups is to describe the symmetries of a system, encoded in the group structure.

One of the key applications of Lie groups is in geometry, where they have been widely applied to study the symmetries of geometric objects. For instance, Euclidean spaces and spheres admit Lie group actions that preserve their geometric properties. The classification of Lie algebras and their representations has led to significant progress in geometry and physics. Lie algebras are extensions of Lie groups, and encode the infinitesimal symmetries of Lie groups.

Lie groups have also found applications in the field of differential equations. The symmetry group of a differential equation characterizes the transformations that preserve the form of the equation. The symmetry group of a differential equation can be used to construct solutions using the method of Lie symmetries. This method has been successfully applied to various classes of differential equations, including linear and nonlinear partial differential equations.

Representation theory is another field where Lie groups have made significant contributions. Lie groups and their algebras bring a natural setting for studying the representation theory of finite and infinite dimensional systems. The explore of representations of Lie groups and algebras has led to numerous results in areas such as harmonic analysis, number theory, and quantum physics.

In addition to the applications mentioned above, Lie groups have also appeared in number theory, probability theory, and statistical physics. Lie groups offer a natural language for describing symmetries in these fields, leading to a better understanding of their structure and behavior.

3.2. Lie Groups future development

Lie groups, as highly integrated and combined discipline in mathematics, are also very diverse in their future development, and can be used in geometry, algebra, and analysis in mathematics, as well as in deep learning in computers, and in nuclear and particle physics in physics. The following are some of the larger branches:

1. Algebraic Geometry: In algebraic geometry, Lie groups play a foundational role in studying moduli spaces of algebraic varieties. Moduli spaces classify algebraic objects (e.g., vector bundles, curves) that share specific properties. Lie groups often act on these spaces, providing symmetry and structure. A fundamental example is the moduli space MC of stable vector bundles over a compact Riemann surface C. The dimension of MC is given by the Riemann-Roch theorem:

$$\dim MC = (n2 - 1)(g - 1),$$

where n is the rank of the vector bundle, and g is the genus of C.

Using Geometric Invariant Theory (GIT), the action of a Lie group G on an algebraic variety X leads to a quotient:

$$X//G,$$

which parametrizes G-orbits of stable points. For example, in the construction of moduli spaces of vector bundles, the gauge group G acts on the space of connections A:

DOI: 10.54254/2753-8818/83/2025.20240

$$M = A/G.$$

2. Homotopy Theory: Hermann Weyl, in The Classical Groups: Their Invariants and Representations**[1]**, first suggested how the topology and algebraic structure of Lie groups could be combined and applied to homotopy theory:

Theorem: Let G be a compact, simply-connected Lie group. Then its cohomology ring H* (G;Z) is isomorphic to an exterior algebra:

$$H^*(G;\mathbb{Z}) \cong \bigwedge(u_1, u_2, \ldots, u_k)$$

where ui are generators of odd degrees determined by the Lie group G.

Hermann Weyl's work provided the basis for Bott periodicity, and Bott combined the topology of Lie groups with homotopy theory in The Stable Homotopy of the Classical Groups[8].

Theorem: The homotopy groups of compact Lie groups exhibit Bott periodicity: $\pi_k(G) \cong \pi_{k+8}(G)$

where G is a compact Lie group and πk (G) denotes its k-th homotopy group.

3. Non-Commutative Geometry: Non-commutative geometry provides a natural generalization of Lie groups by replacing the classical notion of "points" with algebraic structures. In Alain Connes' framework**[9]**, tools such as spectral triples, and quantum groups are central to exploring and describing non-commutative spaces.

Connes defined Spectral Triples as:

$$(A, H, D)$$

where:

- A: A non-commutative algebra representing the "function algebra" of a noncommutative space.
- H: A Hilbert space on which the algebra A acts.
- D: A Dirac operator, a self-adjoint operator that captures the geometry of the space.

Let G be a Lie group that acts on the algebra A and the Hilbert space H. This action is described by:

$$\Phi : G \to \mathrm{Aut}(A)$$

where:

$$\Phi(g)(a) = \alpha_g(a), \quad g \in G, a \in A.$$

The Lie algebra g of G acts on the Dirac operator D via its generators LX as: $[L_X, D] \neq 0, \quad \forall X \in g$.

In non-commutative geometry, quantum groups (Hopf algebras) extend the actions of Lie groups. Hopf algebras satisfy the following axioms:

Co-algebra Structure:

$$\Delta : A \to A \otimes A, \quad \epsilon : A \to C.$$

Antipode:

$$S : A \to A, \quad m \circ (\mathrm{id} \otimes S) \circ \Delta = \epsilon \cdot \mathrm{id}.$$

4. Applications in Machine Learning: In the paper Group Equivariant Convolutional Networks (G-CNNs) by Taco S. Cohen and Max Welling [11], the authors present an innovative approach to convolutional neural networks that integrates Lie group symmetries. This enables the models to become equivariant under transformations associated with specific groups, such as translations and rotations, providing a distinct advantage over traditional convolutional methods.

G-CNNs generalize traditional convolutions by incorporating Lie group structures, enabling equivariance beyond translations. The convolution operation in G-CNNs is redefined as:

DOI: 10.54254/2753-8818/83/2025.20240

$$[f * \psi](g) = \int_X f(x)\psi(g^{-1} \cdot x)\, dx$$

where:

- g is an element of a Lie group G;
- $g \cdot x$ denotes the action of the group G on the input space X (e.g., rotations or translations);
- $\psi(g^{-1} \cdot x)$ is the convolution kernel transformed under the group action.

This extension ensures that G-CNNs remain **equivariant** under the group G, meaning that if the input signal transforms under a group action, the output will transform correspondingly:

$$[f * \psi](g \cdot x) = g \cdot [f * \psi](x).$$

If each layer of the G-CNN satisfies G-equivariance, the entire convolutional network will preserve this property. Mathematically, the equivariance in the l + 1-th layer is expressed as:

$$h_{l+1}(g) = \sigma\left(\int_G h_l(g')\psi(g^{-1}g')\, dg'\right)$$

where:

- $h_l(g)$ is the feature map in the l-th layer;
- $\psi(g^{-1}g')$ is the group-equivariant convolution kernel;
- σ is the non-linear activation function.

This architecture allows G-CNNs to process data with inherent group symmetries efficiently, reducing the need for excessive data preprocessing or augmentation.

The inclusion of Lie groups in G-CNNs addresses several limitations of traditional CNNs:

- **Invariance to Transformations:** By leveraging group structures such as rotations, reflections, and scalings, G-CNNs inherently handle these transformations without requiring extensive data augmentation.
- **Reduced Model Complexity:** Symmetry properties of Lie groups reduce the number of parameters, improving computational efficiency.
- **Improved Generalization:** G-CNNs generalize better across unseen data due to their equivariance properties, especially in tasks like image and pattern recognition.

5. Quantum Computing: Hermann Weyl's work [7] laid the foundation for the integration of Lie groups into quantum mechanics, which has been instrumental in designing quantum gates (such as rotation gates and symmetry-preserving gates). In quantum computing, Lie groups' representations are utilized to design symmetry-preserving quantum gates. For example, rotation gates Rx , Ry , Rz are based on the rotation group SO(3) [10].

1. Rotation Gates:

$$Rx(\theta) = e^{-i\frac{\theta}{2}X}, \quad R_y(\theta) = e^{-i\frac{\theta}{2}Y}, \quad Rz(\theta) = e^{-i\frac{\theta}{2}Z},$$

where X,Y,Z are the Pauli matrices.

Quantum gates are implemented using the exponential map from the Lie algebra to the Lie group: $U = e^{-iHt}$, where H is the system's Hamiltonian.

The above equations demonstrate how Lie groups describe quantum state evolution geometrically, connecting quantum gate design to the symmetry properties of Lie groups.

The symmetric properties of Weyl groups have been utilized in quantum computing for entanglement construction. Specifically, the representation theory of the SU(2) group directly corresponds to single-qubit operations.

DOI: 10.54254/2753-8818/83/2025.20240

In the Bloch sphere representation, a single-qubit state is described as [10]:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle,$$

where θ and ϕ are spherical coordinates representing the state on the Bloch sphere. The Bloch sphere's structure is governed by the Lie group SU(2), which is the group of all 2 × 2 special unitary matrices with determinant 1. SU(2) acts on the state space, defining rotations and transformations of quantum states.

An arbitrary rotation can be expressed as [10]:

$$U(\theta,\phi) = e^{-i\phi Z}e^{-i\theta Y}e^{-i\gamma Z}.$$

The operation U(θ,ϕ) is an element of the Lie group SU(2), which describes all unitary transformations that preserve the quantum state's norm.

4. Conclusion

Lie groups are a testament to the profound interconnectedness of mathematics, physics, and computational sciences. Their historical evolution, from Sophus Lie's pioneering work on continuous transformation groups to modern advancements in non-commutative geometry, quantum mechanics, and machine learning, highlights the rich and diverse origins of these structures.

Lie groups originated as tools to describe continuous symmetries in geometry and analysis. Starting with Augustin-Louis Cauchy's groundwork on continuous groups, Sophus Lie formalized these ideas into what we now recognize as Lie groups. The contributions of Felix Klein and Hermann Weyl later expanded their scope, integrating Lie groups into the domains of topology, algebra, and quantum physics. This synthesis underscores their foundational role in mathematics as unifiers of geometry, analysis, and algebra.

The continued development of Lie groups promises new insights and applications. Their integration into deep learning, quantum computing, and advanced geometry reflects the ongoing evolution of this field. As Lie groups bridge theoretical and applied disciplines, their capacity to unify and innovate will undoubtedly expand.

In conclusion, Lie groups are not merely a mathematical construct but a dynamic and evolving tool, linking historical developments with future possibilities. In the future, with the deepening of the Langlands program and the increase of interdisciplinary research in mathematics, Lie groups, which embody mathematical analysis, abstract algebra, linear algebra, and topology, will serve as a fundamental tool across a growing number of fields.

References

[1] *Hermann Weyl, The Classical Groups: Their Invariants and Representations, Princeton University Press, 1928.*

[2] *Louis Cauchy, Cours d'Analyse de l'École Royale Polytechnique, 1821.*

[3] *Felix Klein, Vorlesungen ¨uber Nicht-Euklidische Geometrie, Teubner, 1872.*

[4] *Andr´e Weil, L'int´egration dans les groupes topologiques etses applications, First Edition: 1940, Second Edition: 1951.*

[5] *Sophus Lie and Friedrich Engel, Theorie der Transformationsgruppen, Teubner, Leipzig, 1888.*

[6] *Jo¨el Merker, Theory of Transformation Groups (Vol. I) , Modern Presentation of Lie's Work (1888), Springer, 2010.*

[7] *Hermann Weyl, Gruppentheorie und Quantenmechanik, Hirzel, Leipzig, 1928.*

[8] *Bott, R. The Stable Homotopy of the Classical Groups. Annals of Mathematics, 1959.*

[9] *Connes, A. Noncommutative Geometry. Academic Press, 1994.*

[10] *Michael A. Nielsen and Isaac L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2010.*

[11] *Cohen T. S. and Welling M. , Group Equivariant Convolutional Networks, 2016.*