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Frequency Analysis of Impulse Response in Linear Damped Harmonic Oscillators: A Comparative Study of Fourier and Laplace Transforms with Numerical Validation

Yue Liu

Tandon School of Engineering, New York University, New York, USA
yl11518@nyu.edu

Abstract: Analyzing a system's response is a crucial topic in various fields such as electric engineering, quantum mechanics, and applied mathematics. Analyzing systems from frequency domain is a promising and important approach. Aside from the widely accepted numerical method to solve for the displacement function of a linear damped harmonic oscillator driven by an impulsive force, various mathematical transforms have been developed to provide new approaches. Two transforms – Fourier Transform and Laplace Transform– are examined for their accuracy and efficiency. They are used to compute the responses of three different impulses – finite rectangular impulse, Gaussian impulse, and exponentially decaying impulse – and compared to the accurate answer obtained by the numerical approach. The result states that none of the two transforms can handle all three impulses, and different transforms have different strengths and weaknesses. Overall, this paper proves the validity of using mathematical transforms to solve for linear systems and explores the limitations of Fourier Transform and Laplace Transform. This paper is also considered to be a useful and comprehensive guide for introductory readers who are interested in frequency analysis. Future research can focus on investigating the responses for different impulses using different transforms.

Keywords: Fourier Transform, Laplace Transform, Impulse Response, Linear Damped Harmonic Oscillator.

1. Introduction

Damped harmonic oscillator is a typical model of a phenomenon that exists nearly everywhere in physical life. Well known examples include mass attached to a spring in a viscous fluid (also called duffing oscillator), RLC circuit, ocean wave, and even doors with soft-closing mechanisms.

The key difference between linear damped harmonic oscillator (DHO) and linear simple harmonic oscillator (SHO) is that DHO will gradually decrease to equilibrium (static state) from motion, while SHO will oscillate forever. This implies that there must have been a driven force acting on the oscillator for motion to be observed. Most of such driven forces are short lived or rapidly decaying and known as impulsive forces.

Therefore, analyzing responses to various driven forces of a linear damped harmonic oscillator is a crucial part in classical mechanics. One widely accepted and used method to solve the displacement of the oscillator with respect to time is the numerical method (directly solving the differential

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equation), and it is usually considered to be most direct and universally applicable. Such numerical method only focuses on the physical space (displacement of the oscillator over time). However, with more and more mathematical transforms being developed, analyzing such a motion from the frequency space becomes feasible. Such transforms include the renowned Fourier Transform and Laplace Transform.

This paper primarily focuses on the comparison between the numerical method, Fourier Transform and Laplace Transform. Basic theories, equations and the general procedures of computing are explained first. The responses to three different types of impulses – finite rectangular impulse, Gaussian impulse, and exponentially decaying impulse – are analyzed for each transform. The results of three methods are compared to investigate each transform's advantages and disadvantages.

This paper provides a thorough investigation of general theories and approaches of using mathematical transforms to analyze linear damped harmonic oscillations. It can be a good reference for introductory level scholars to gain an intermediate understanding of the theories and methodologies in relevant fields. Further research can focus on other similar oscillations or real-world phenomena that may introduce mathematical transforms as alternative approaches.

2. Literature review

Using frequency analysis to solve systems' responses is a promising method. Fourier Transform and Laplace Transform are two of the transforms that are widely used. Fourier Transform is usually used in investigating steady state behavior or long-term periodic driven force; on the other hand, Laplace Transform is better in dealing with non-periodic driven force and transient signals with non-zero initial conditions.

Many contemporary papers have introduced the methodologies and results of analyzing different systems using frequency analysis. For example, Muhammad et al [1], Tang et al [2], Turkosz [3], and [4] Rosenfelder explored properties of Fourier Transform to investigate responses of their particular systems. Among those, Rosenfelder focuses on the structure function of damped harmonic oscillator, which is relevant to the system model used in this paper. Similarly, Lu et al [5] and Alharthi et al [6] applied Laplace Transform to their own research.

However, unlike previous detailed yet complicated studies, this paper introduces a much direct and more comparative analysis suitable for introductory scholars. More specifically, this paper focuses on detailed explanations of fundamental knowledge and techniques of applying mathematical tansforms. Likewise, Kovacs' work on explaining how to apply Laplace Transform for the damped driven oscillator is also worth referring to [7]. *Control Systems Engineering* by Nise [8] and *Signals and Systems* by Oppenheim et al [9] are referred during the research process of this paper for essential foundational knowledge and applications in Fourier and Laplace Transform, and they are highly recommended for readers who seek deeper understandings in frequency analysis.

3. Methodologies

The derivation of the system's characteristics and the numerical approach is a generalization of Chapter 3 from page 108 to 136 in the book *Classical Dynamics of Particles and Systems* by Thorton and Marion [10].

The general ODE (ordinary differential equation) for linear damped harmonic oscillators is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m} \tag{1}$$

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where $\beta \equiv \frac{b}{2m}$ is the damping constant and $\omega_0 \equiv \sqrt{\frac{k}{m}}$ is the natural angular frequency. The derivation of this equation is not shown in this paper but discussed in great detail by Thorton and Marion from page 108 to 109 [10] and in numerous literatures as well.

3.1. General way of solving ODE

Since the differential equation is linear, the final response $x(t)$ can be decomposed into the summation of the homogeneous solution and particular solution:

$$x(t) = x_h(t) + x_p(t) \tag{2}$$

3.1.1. Particular solution

The particular solution is the solution that can satisfy equation (1) with right-hand side is the driven force. Usually, one can guess a solution, also known as ansatz, in the form of driven force with unknown coefficients, and then plug the ansatz into equation (1) to solve for the coefficients for the particular solution.

3.1.2. Homogeneous solution

The homogeneous solution $x_h(t)$ is obtained by directly solving equation (1) but disregarding the driven force (or the right-hand side):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{3}$$

The detailed derivation of the results has been sufficiently explained in countless literatures and textbooks. Define $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ .The general solution is given by:

Underdamping: $\omega_0 > \beta$

$$x_c(t) = e^{-\beta t}(A_1 cos\omega_1 t + A_2 sin\omega_1 t) \tag{4}$$

Critically damping: $\omega_0 = \beta$

$$x_c(t) = (A + Bt)e^{-\beta t} \tag{5}$$

Overdamping: $\omega_0 < \beta$, define $\lambda \equiv \sqrt{\beta^2 - \omega_0^2}$

$$x_c(t) = e^{-\beta t}\left(A_+ e^{\lambda t} + A_- e^{\lambda t}\right) \tag{6}$$

Combining the homogeneous solution with the particular solution according to equation (2) to obtain the total response, and then plug it into equation (3) and evaluate the initial conditions to get the values of coefficients A's and B.

3.1.3. Numerical method

For any linear damped harmonic oscillator, the homogeneous solution is always in one of the forms shown above depending on damping conditions. Nevertheless, the form of the particular solution can be varied according to countless different forms of driven forces. In some cases, the driven force can be too arbitrary and complicated to guess an ansatz, as introduced in section 3.1.1. Then, the numerical method may be used to resolve the issue. Since any arbitrary force can be decomposed into a summation (or integration) of infinitely many unit impulses, the response of the system can be the combined effect of all the individual responses to those unit impulses.

Therefore, the general approach is as follows:

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1. Obtain the response of a unit impulse function at equilibrium position (zero initial condition).
2. Integrate the responses of all the impulses that add up to be the arbitrary driven force to get the partial response with zero initial conditions.
3. Solve the response due to the initial condition using the homogeneous solution and combine it with the partial response to get the total response.

The math of this approach is fairly complicated and is neglected in this paper. One can refer to Thornton and Marion's book from page 129 to page 136 for more information. However, the general formula is, thankfully, not complicated. The formulas for the three different damping cases are different. For simplicity, this paper only evaluates the case of underdamping, which is given by:

$$x(t) = \int_{-\infty}^{t} \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]\, dt' \tag{7}$$

where t' is the integration parameter. The lower limit in general is $-\infty$, but it is always true to replace it by the time when the force is just applied (usually is set to be 0).

Keep in mind that equation (7) gives the response under the assumption of zero initial condition. Since the system is linear, the response due to the initial condition can be calculated separately using equation (4) and be added to the result obtained by equation (7).

3.2. Using Fourier Transform

The numerical approach analyses the system from the time domain. Transforms like Fourier Transform and Laplace Transform can bring a new view from the frequency domain.
The definition of Fourier Transform is given by:

$$\mathcal{F}[\phi(t)] = \Phi(\lambda) = \int_{-\infty}^{\infty} \phi(t) e^{-i\lambda t} dt \tag{8}$$

$\mathcal{F}$ is the transform operator, $\phi(t)$ is the response in the time domain and $\Phi(\lambda)$ is the response in the frequency domain where λ is the frequency variable.

Simply speaking, Fourier Transform converts a function with respect to time to another function with respect to frequency. The reason we do the transformation is that, compared to the numerical method discussed in the last part, solving the response ODE (equation (1)) in the frequency domain is much easier. Linearity of transforms guarantees that all the computations done in the frequency domain will be mathematically equivalent to corresponding computations done in the time domain. So, if the transformed response is solved in the frequency domain, the response in time domain can be obtained by simply applying the inverse transform:

$$\mathcal{F}^{-1}[\Phi(\lambda)] = \phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\lambda) e^{i\lambda t} d\lambda \tag{9}$$

The reason that solving the ODE is easier in the frequency domain has to do with the mathematical properties of Fourier Transform:

$$\mathcal{F}\left[\phi^{(n)}(t)\right] = (i\lambda)^n \mathcal{F}[\phi(t)] \tag{10}$$

This means that Fourier Transform can reduce a higher-order ODE to a zeroth-order equation.

Apply the property from equation (10) to equation (1), one can generalize a transfer function:

$$H_F(\lambda) = \frac{X}{G} = \frac{1}{-\lambda^2 + i2\beta\lambda + \omega_0^2} \tag{11}$$

where $G = \mathcal{F}\left[\frac{F}{m}\right]$. The transfer function gives the ratio between the output X and the input G for this system in the frequency domain. This means that, given any input G, GH_F gives the corresponding response X.

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Therefore, one can solve for $x(t)$:

$$x(t) = \mathcal{F}^{-1}[X] = \mathcal{F}^{-1}[GH_F] \tag{12}$$

In summary, the general approach to solve the system is:

1. Take the Fourier Transform of the input and obtain G.
2. Use $X = GH_F$ to obtain X.
3. Take the inverse transform of X to get the response in the time domain.

Some may have noticed that this approach using Fourier Transform disregards the damping conditions. Indeed, this is another advantage of using transforms.

3.3. Using Laplace Transform

Laplace Transform follows the same logic as Fourier Transform.
The definition of Laplace Transform is given by:

$$\mathcal{L}[k(t)] = K(s) = \int_{0-}^{\infty} k(t)e^{-st}dt \tag{13}$$

The inverse transform is given by:

$$\mathcal{L}^{-1}[K(t)] = k(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} K(s)\, e^{st}ds \tag{14}$$

where s is a complex number. σ is the real part of s and is carefully selected to be greater than all poles and zeros to ensure the convergence.

Different from Fourier Transform, Laplace Transform of derivatives of a function is in another form:

$$\mathcal{L}\left[k^{(n)}(t)\right] = s^n K(s) - \sum_{k=1}^{n} s^{n-k}k^{(k-1)}(0) \tag{15}$$

It seems to be more complicated than Fourier Transform. However, one of the advantages of Laplace Transform is that it takes initial conditions into account, while Fourier Transform does not.
Apply Laplace Transform to equation (1) and eventually gets:

$$\frac{X}{G} = \frac{1}{s^2+2\beta s+\omega_0^2} + \frac{(s+2\beta)x(0)+\dot{x}(0)}{(s^2+2\beta s+\omega_0^2)\cdot G} \tag{16}$$

The first term is the general transfer function of zero initial condition, and the second term evaluates the response due to the initial condition:

$$H_L(s) = \frac{1}{s^2+2\beta s+\omega_0^2},\ C_L = \frac{(s+2\beta)x(0)+\dot{x}(0)}{(s^2+2\beta s+\omega_0^2)} \tag{17}$$

And the transformed response is:

$$X = GH_L + C \tag{18}$$

The final response in the time domain is:

$$x(t) = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[GH_L + C] \tag{19}$$

Similar to Fourier Transform, Laplace Transform doesn't care about the damping conditions as well, but it evaluates the initial conditions.

3.4. Results for different responses

In this part, different forms of impulses are analyzed using the three methods explained above.

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3.4.1. Rectangular impulse

First start with a rectangular impulse, which is one of the simplest forms of impulse. Essentially, rectangular impulse is a step function with constant force applied on the system for a finite amount of time:

$$F(t) = a, \ \ 0 < t < t_0 \tag{20}$$

Assign values for the parameters of the system for the sake of comparison:

$$\begin{cases} F = 100, \ \ 0 < t < 1 \\ m = 1 \\ b = 2 \ (\beta = 1) \\ k = 10 \\ x(0) = \dot{x}(0) = 0 \end{cases} \tag{21}$$

3.4.1.1. Numerical method

In general, the response given by the numerical method is in the form of:

$$x(t) = \begin{cases} \int_0^t \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \, sin[\omega_1(t-t')] \, dt', \ \ 0 < t < t_0 \\ \int_0^{t_0} \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \, sin[\omega_1(t-t')] \, dt', \ \ \ t_0 < t \end{cases} \tag{22}$$

Plug in the values given by equation set (21), the final response is given by:

$$x(t) = \begin{cases} 10 - \frac{10}{3e^t}(3\cos(3t) + \sin(3t), & 0 < t < 1 \\ \frac{10}{3e^t}(3ecos(3-3t) - 3\cos(3t) + esin(3t-3) - \sin(3t)), & t > 1 \end{cases} \tag{23}$$

3.4.1.2. Fourier Transform

The Fourier Transform of a finite rectangular impulse in general is:

$$\mathcal{F}[g] = G = \int_{-\infty}^{\infty} g e^{-i\lambda t} \, dt = \int_0^{t_0} g e^{-i\lambda t} \, dt = \frac{g}{-i\lambda} e^{-i\lambda t} \Big|_0^{t_0} = \frac{ig}{\lambda}\left(e^{-i\lambda t_0} - 1\right) \tag{24}$$

Multiply by H_F to get X:

$$X = GH_F = \frac{\frac{ig}{\lambda}(e^{-i\lambda t_0} - 1)}{-\lambda^2 + i2\beta\lambda + \omega_0^2} \tag{25}$$

Now plug in the values from equation set (21) and get:

$$X = \frac{\frac{i100}{\lambda}(e^{-i\lambda} - 1)}{-\lambda^2 + i2\lambda + 10} = \frac{i100(e^{-i\lambda} - 1)}{-\lambda^3 + i2\lambda^2 + 10\lambda} \tag{26}$$

However, it turns out that it is too complicated to take the inverse Fourier Transform to get the actual response in the time domain. This also reflects the mathematical nature of Fourier Transform that is an extrapolation of Fourier Series, which only applies to continuous and periodic functions. Therefore, Fourier Transform can be not useful to analyze exceptionally non-periodic functions such as the finite rectangular function in this case. More relevant discussion is included in the discussion section.

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3.4.1.3. Laplace Transform

The Laplace Transform of a finite rectangular impulse in general is:

$$\mathcal{L}[g] = G = \int_0^{\infty} ge^{-st}dt = \int_0^{t_0} ge^{-st}dt = -\frac{g}{s}e^{-st}\Big|_0^{t_0} = -\frac{g}{s}(e^{-st_0} - 1) \tag{27}$$

Then, find H_L and C_L:

$$GH_L = -\frac{g(e^{-st_0}-1)}{s^3+2\beta s^2+s\omega_0^2}, C_L = \frac{(s+2\beta)x(0)+\dot{x}(0)}{(s^2+2\beta s+\omega_0^2)} \tag{28}$$

Since the initial condition is equilibrium, C_L is automatically zero. One can get:

$$X = GH_L + C_L = -\frac{100(e^{-s}-1)}{s^3+2s^2+10s} + 0 = \frac{100-100e^{-s}}{s^3+2s^2+10s} \tag{29}$$

Then, take the inverse Laplace Transform of X to obtain the response in the time domain, $x(t)$:

$$x(t) = \mathcal{L}^{-1}[X] = 10 - 10e^{-t}\cos(3t) - \frac{10}{3}e^{-t}\sin(3t) - \left(10 - 10e^{-(t-1)}\cos(3(t-1)\right) - \frac{10}{3}e^{-(t-1)}\sin(3(t-1)) \tag{30}$$

However, this is not the final response for $t > 0$. The problem comes from the fact that all the terms that include $(t-1)$ are not valid in the time interval $0 < t < 1$. Another way to think of this is to decompose the impulse function into the difference of two step functions, and the second step function should be introduced to the system only after $t = 1$.
Therefore, equation (30) is for $t > 1$. The response for $t < 1$ is exactly the first half of equation (30). So, the final response given by Laplace Transform is:

$$x(t) = \begin{cases} 10 - 10e^{-t}\cos(3t) - \frac{10}{3}e^{-t}\sin(3t), & 0 < t < 1 \\ \left(10 - 10e^{-t}\cos(3t) - \frac{10}{3}e^{-t}\sin(3t) - \left(10 - 10e^{-(t-1)}\cos(3(t-1)) - \frac{10}{3}e^{-(t-1)}\sin(3(t-1))\right)\right), & 1 < t \end{cases} \tag{31}$$

3.4.1.4. Comparison

Plotting equation (23) and equation (31) gives graphs for the response using two different methods:

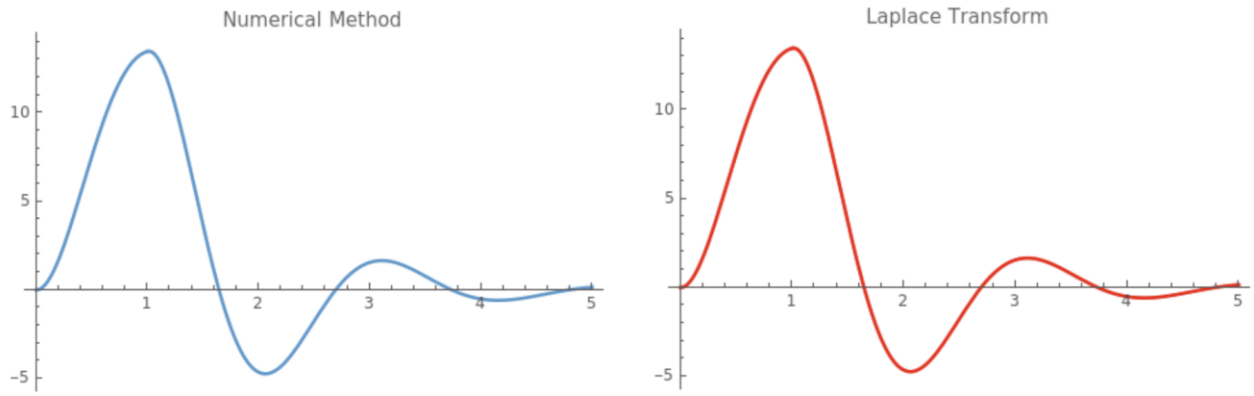


Figure 1: Comparison plot of numerical method and Laplace Transform method in solving for the response of a finite rectangular impulse

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One can clearly tell that the two graphs are nearly identical. This shows that Laplace Transform successfully gets the correct answer. On the other hand, Fourier Transform faces tremendous computation, which is determined to be not useful at all in handling finite rectangular impulse.

3.4.2. Gaussian impulse

Next, a Gaussian impulse evaluated:

$$F(t) = Ee^{-at^2}, t > 0 \tag{32}$$

where E is a constant.

Similar to the last case but with *non-zero initial conditions*, parameter values are given by:

$$\begin{cases} E = 100, \ \ 0 < t \\ m = 1 \\ b = 2 \ (\beta = 1) \\ k = 10 \\ x(0) = 10 \\ \dot{x}(0) = 5 \\ a = 1 \end{cases} \tag{33}$$

3.4.2.1. Numerical method

The general form of the response for *zero initial condition* is still in the form given by equation (22). Plugging in values from equation set (33) yields the response:

$$x_1(t) = \frac{e^{-1-\frac{3}{2}i} \cdot e^{-(1+3i)t} \cdot \sqrt{\pi}}{12} \cdot \left[\begin{array}{l} ie^{3i}\left(\mathrm{Erf}\left(\frac{1}{2}+\frac{3i}{2}\right) - \mathrm{Erf}\left(\frac{1}{2}+\frac{3i}{2}-t\right)\right) \\ +e^{6it}\left(-\mathrm{Erfi}\left(\frac{3}{2}+\frac{i}{2}\right) + \mathrm{Erfi}\left(\frac{3}{2}+\frac{i}{2}-it\right)\right) \end{array} \right] \tag{34}$$

where "Erf" is the error function and "Erfi" is the imaginary error function:

$$\begin{cases} \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\ \mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt \end{cases} \tag{35}$$

However, this case involves *non-zero initial conditions*: $x(0) = 10$ and $\dot{x}(0) = 5$. Using equation (4), the general solutions for coefficients for A_1 and A_2 are given by:

$$A_1 = x(0), A_2 = \frac{\dot{x}(0) + \beta x(0)}{\omega_1} \tag{36}$$

So, the response due to the initial conditions is given by:

$$x_2(t) = e^{-t}(10\cos(3t) + 5\sin(3t)) \tag{37}$$

and the total response is:

$$x(t) = x_1(t) + x_2(t) \tag{38}$$

3.4.2.2. Fourier Transform

Following the same procedure as before, one can get:

$$X = GH_F = \frac{\frac{E}{\sqrt{2a}} \cdot e^{-\frac{\lambda^2}{4a}}}{-\lambda^2 + i2\beta\lambda + \omega_0^2} = \frac{\frac{100}{\sqrt{2}} e^{-\frac{\lambda^2}{4}}}{-\lambda^2 + i2\lambda + 10} \tag{39}$$

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Nevertheless, it turns out that the inverse of X is still not tractable.

3.4.2.3. Laplace Transform

The response in the frequency domain can be computed as follows:

$$X = \frac{25\sqrt{\pi}\cdot e^{\frac{s^2}{4}}}{s^2+2s+10} + \frac{10s+25}{s^2+2s+10} \tag{40}$$

However, it turns out that the inverse Laplace Transform of X is not tractable as well.

3.4.2.4. Comparison

For Gaussian impulse, both Fourier Transform and Laplace Transform fail to perform the task.

3.4.3. Exponentially decaying impulse

It seems that the second power of t in Gaussian impulse poses challenges for transforms to mathematically work out. Instead, in this section an exponential impulse is investigated. Exponentially decaying impulse is one of the most common driven forces in real life, which is given by:

$$F(t) = Ee^{-at}, t > 0 \tag{41}$$

Note that the t is now to the first power. The values of parameters in equation set (33) are still used in this section.

3.4.3.1. Numerical method

Since the system has a non-zero initial condition as well, the numerical method is first used to evaluate the response to the impulse in case of zero initial condition:

$$x_1(t) = \frac{200}{9}\cdot\frac{\sin^2(\frac{3t}{2})}{e^t} \tag{42}$$

The response due to initial condition should be the same as equation (37).
Combine both responses to have the final response:

$$x(t) = \frac{200}{9}\cdot\frac{\sin^2(\frac{3t}{2})}{e^t} + e^{-t}(10\cos(3t) + 5\sin(3t)) \tag{43}$$

3.4.3.2. Fourier Transform

Fortunately, the inverse Fourier Transform is tractable in this case:

$$x(t) = Re(\mathcal{F}^{-1}[X]) = Re\left[-\frac{50}{9}\cdot\frac{(1-e^{3it})^2}{e^{(1+3i)t}}\right] = \frac{200}{9}\cdot\frac{\sin^2\left(\frac{3t}{2}\right)}{e^t} \tag{44}$$

which is identical to equation (42) that is the response for zero initial condition. One more step is to add the response due to the initial condition given by equation (37) to get the final response, which is the same as equation (43).

3.4.3.3. Laplace Transform

The final response is also tractable for Laplace Transform:

$$x(t) = Re(\mathcal{L}^{-1}[X]) = \frac{5}{18}\cdot e^{-t}(40 - 4\cos(3t) + 18\sin(3t)) \tag{45}$$

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By manipulating trigonometric identities, one can prove that equation (45) is identical to equation (43).

3.4.3.4. Comparison

The plot for all three methods is shown below:

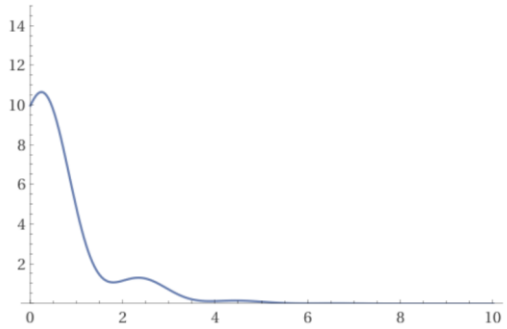


Figure 2: Plot of the response of an exponentially decaying impulse solved by using all three methods

This section proves that all three methods can solve for the response of the system driven by an exponentially decaying force. One may have noticed that taking inverse Laplace Transform of X directly yields the whole response, whereas Fourier Transform needs to add the response due to initial conditions. This reflects what has been discussed in section 3.3 that Fourier Transform neglects the response due to initial conditions.

4. Discussions

Based on the three different impulses analyzed, it is true that frequency analysis is a promising approach in solving for linear damped harmonic oscillating system. However, their applicability significantly varies across different impulse types.

For finite rectangular impulse, Laplace Transform does an excellent job. Nevertheless, Fourier Transform shows its insufficiency in handling such a simple case. This is due to the mathematical nature of Fourier Transform. Fourier Transform is an extension from Fourier Series, which is used to approximate infinite periodic functions. Therefore, though Fourier Transform is widely used in frequency analysis, its fundamental logic is still based on analyzing a periodic function but with infinitely long period. That is why it is often extremely hard and even not possible to perform the inverse transform of the response in frequency space for non-periodic impulses. However, Fourier Transform can be exceptionally useful to analyze the steady state of a system driven by a periodic force, which is not the focus of this paper.

The reason for Laplace Transform to successfully handle the case for finite rectangular impulse is that Laplace Transform is defined without the assumption of periodicity. Nevertheless, Laplace Transform cannot be tractable when analyzing Gaussian impulse. Like Fourier Transform, Laplace Transform also faces the difficulty of transforming the response in frequency domain back to the time domain due to computational limitation.

Therefore, since transforms are not universally applicable, practitioners should choose transforms accordingly and properly. Otherwise, it may result in complicated computations and waste of time.

Keep in mind that this paper only investigates the underdamped conditions. Though all damping conditions appear to be in the same form in the frequency domain, it is still worth investigating potential differences that may be caused by different damping conditions.

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Please note that this paper only focuses on three different impulses. The results cannot be generalized to all arbitrary impulses. Further investigations can focus on the transient responses of other typical impulses that are common in life, such as a bell-shaped impulse, sinusoidal impulse (and the steady state), and delta impulse, to more comprehensively explore the usefulness and different advantages of various mathematical transforms.

5. Conclusions

This paper discusses the validity of using Fourier Transform and Laplace Transform in solving for the transient response of linear harmonic damped oscillator driven by different impulses. Conclusively, using Fourier Transform or Laplace Transform can solve the response, but none of them can handle all the cases. This suggests that different mathematical transforms display unique strengths and weaknesses. For particular impulses, using transform can be unexpectedly easy (such as using Laplace Transform to analyze finite rectangular impulse). Therefore, it is suggested to use particular mathematical transform (or even invent new transforms) to reduce a complicated system that is hard to solve in time domain to a simpler case in the frequency domain.

Future investigations can be done in exploring how to invent mathematical transforms with targeted features and properties. Existing investigations have started to develop new and specialized mathematical transforms tailored to specific systems' characteristics and analytical needs. This approach excellently resolves the issue that certain existing and well-known transforms may not be applicable to all cases. However, it requires profound mathematical skills to develop such new transforms, and the methods can be quite different across cases. If such a way can be generalized, then it will provide tremendous help in all kinds of frequency analysis and system analysis.

Lastly, one of the important purposes of this paper is to provide introductory scholars or students with an overview of solving a system from frequency domain. Relevant knowledge regarding frequency analysis is explained. So, this paper is expected to serve as a guide for those seeking a quick understanding in frequency analysis.

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References

[1] *Niaz Muhammad, Tahira & Hameed, Attia & Ramzan, Madiha & Khalid, Muhammad. (2020). Impulsive Response of Damped Harmonic Oscillator Via Fourier Transform. Nature and Science. 18. 70-81. 10.7537/marsnsj181220.10.*

[2] *Tang, Z., Dong, Y., Liu, H., & Li, Z. (2022). Frequency Domain Analysis Method of Tuned Liquid Damper Controlled Multi-Degree of Freedoms System Subject to Earthquake Excitation. Journal of Building Engineering, 48, 103910.*

[3] *Turkosz, D. J. (2010). Transient Structural Analyses Using Frequency-Domain Computations and Fourier Transform Techniques (Master's thesis). The Pennsylvania State University.*

[4] *Rosenfelder, R. (2003). Structure Function of a Damped Harmonic Oscillator. Physical Review C, 68(3), 034602.*

[5] *Lu, H., Fan, T., Chen, C., & Ma, Y. (2022). Rapid Decoupling Laplace-Domain Algorithm for Dynamic Response Estimation of Offshore Structures with Asymmetric System Matrices. Ocean Engineering, 246, Article 110534.*

[6] *Alharthi, N., Atangana, A. & Alkahtani, B. (2023). Analytical and Numerical Analysis of Damped Harmonic Oscillator Model with Nonlocal Operators. Demonstratio Mathematica, 56(1), 20220230.*

[7] *Kovacs, J. S. (2013). Laplace Transform for the Damped Driven Oscillator (MISN-0-47). Project PHYSNET, Michigan State University.*

[8] *Nise, N. S. (2015). Control Systems Engineering (7th ed.). Wiley.*

[9] *Oppenheim, A. V., Willsky, A. S., & Nawab, S. H. (1997). Signals and Systems (2nd ed.). Prentice Hall.*

[10] *Marion, J. B., & Thornton, S. T. (2004). Classical Dynamics of Particles and Systems (5th ed., pp. 108–136). Brooks Cole.*