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Euler's Formula and Its Applications on Trigonometry and Calculus

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Abstract: Euler's formula, the combination of trigonometric and exponential forms, plays a crucial role in simplifying expressions and solving integrals across mathematics, physics, and engineering. Although its theoretical gravity has been well acknowledged, its practical applications in simplifications and integrations are neglected in the early stages. This paper specializes in reintroducing the applicability of Euler's formula in solving problems, including trigonometric functions, definite integrals, and simple indefinite integrals. This paper aims to demonstrate corresponding examples with various difficulties and rationalize Euler's formula in different forms to solve them clearly. The method of this paper focuses on transforming trigonometric expressions into exponential form to reduce the computational complexity. Hence, this method shows the symmetry in expressions, which helps derive the Fourier series by applying complex exponentials. The final goal of the paper is to reinforce problem-solving skills through the further analysis of Euler's formula and lay the foundation for the upcoming nonlinear partial differential equations.

Keywords: Euler's formula, trigonometric functions, definite integrals, indefinite integrals, Fourier series.

1. Introduction

Euler's formula, $e^{ix} = \cos x + i \sin x$, as known as one of the elementary formulas in mathematics. This formula combines the base of natural logarithms e, the imaginary unit i and trigonometric functions, and has important application value in solving trigonometric functions, definite integrals and indefinite integrals. In addition, under the tight connection between exponential and trigonometric functions, Euler's formula can efficiently simplify convoluted mathematical categories, from complex analysis to Fourier analysis. Furthermore, as an unshakable cornerstone, the significance of Euler's formula has even influenced more profound subjects, including, but not limited to, topology, number theory, and quantum physics.

Over the past few centuries, Euler's formula has highly and widely utilized in various types of studies. In 1757, the outstanding work written by Leonhard Euler himself, for the first time, introduced Euler's formula in the expansion of trigonometric functions and considered 'functions' as the principal objects in the mathematical community [1]. The masterpiece constructed a bridge, leading to the innovation of further mathematics and physics in that contemporary era and the generations [1]. Subsequently, around the mid-19th century, Bernhard Riemann delved into

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nonlinear partial differential equations by exploiting Euler's formula [2]. Building on this foundation, Riemann concentrated on the analysis of shock and rarefaction waves, accelerating the progress of fluid dynamics [2]. Moreover, in 1957, Peter D. Lax profoundly examined the nonlinear phenomenon that existed in Euler's formula, including the formation of shock waves, the framework of entropy admissibility, etc [3].. To summarize the whole mathematical theory around Euler's formula, in 2016, Constantine M. Dafermos published a highly representative paper that rigorously provided theoretical foundations for each category of studies [4]. The achievement has become a crucial part of the present-day theory of Euler's formula. In summary, given the importance of Euler's formula beyond both pure and applied mathematics, it is vital not only to understand the meaning of how to solve it, but to deeply explore the application in a variety of areas.

This paper illustrates the simplification from trigonometry into exponential form and the derivation of the Fourier series through increasingly complex examples. It aims to enhance the ability to solve calculus and build a strengthened platform for further studies.

2. Applications on trigonometric functions

2.1. Theorems

Euler's formula is

$$e^{ix} = \cos x + i \sin x \tag{1}$$

Its complex conjugate is

$$e^{-ix} = \cos x - i \sin x \tag{2}$$

Add equations (1) and (2), it derives

$$e^{ix} + e^{-ix} = (\cos x + i \sin x) + (\cos x - i \sin x)$$

$$e^{ix} + e^{-ix} = 2 \cos x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{3}$$

Subtract (2) from (1), it derives

$$e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x)$$

$$e^{ix} - e^{-ix} = 2\mathrm{i} \sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{4}$$

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These two equations construct the entire framework of trigonometric functions and bond closely to complex exponential functions, providing a foundation of definite and indefinite integrals.

2.2. Examples

2.2.1. Simplify $\sin x \bullet \cos x$

From the theorem, the product of (3) and (4) is

$$\sin x \bullet \cos x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ix}+e^{-ix}}{2}\right) \tag{5}$$

Hence the equation (5) can be expanded into

$$\sin x \bullet \cos x = \frac{1}{4i}\left(e^{ix}+e^{-ix}\right)\left(e^{ix}-e^{-ix}\right) \tag{6}$$

Due to the formula of two squares

$$(a+b)(a-b) = a^2 - b^2 \tag{7}$$

The equation (6) can be converted into

$$\sin x \bullet \cos x = \frac{1}{4i}\left(e^{i2x}-e^{-i2x}\right) \tag{8}$$

From the theorem, it knows

$$\sin 2x = \frac{e^{i2x}-e^{-i2x}}{2i}$$

$$2i\sin 2x = e^{i2x}-e^{-i2x} \tag{9}$$

Replace the bracket in (8), solving

$$\sin x \bullet \cos x = \frac{1}{2}\sin 2x \tag{10}$$

2.2.2. Simplify $\cos^2 x$

From the theorem, it knows (3)，then the equation (3) is squared

$$\cos^2 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 \tag{11}$$

The equation (11) can be expanded into

$$\cos^2 x = \frac{e^{i2x}+2+e^{-i2x}}{4}$$

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$$\cos^2 x = \frac{1}{2} + \frac{1}{4}\left(e^{i2x} + e^{-i2x}\right) \tag{12}$$

From the theorem, it knows

$$\cos 2x = \frac{e^{i2x} + e^{-i2x}}{2}$$

$$2\cos 2x = e^{i2x} + e^{-i2x} \tag{13}$$

Replace the bracket in (12), solving

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \tag{14}$$

The above derivation indicates that mixed and squared trigonometric functions can be simplified into based trigonometric forms using the transformation of Euler's formula. These results lay a groundwork for the solutions of subsequent complex integrals involving trigonometric functions.

3. Applications on definite and indefinite integrals

3.1. Background

The most common application of definite integrals is to calculate the area under a curve or the total cumulative variation in a function within a given interval [5]. The basic model is:

$$Area = \int_a^b f(x)dx \tag{15}$$

Indefinite integrals are often used to find the original function. They are considered as essential tools to derive the differentiations and convert between physical quantities. In addition, the constant C must be added as a part of antiderivatives since the indefinite integral is not seen as an individual function [6]. The basic model is:

$$\int f(x)dx = F(x) + C \tag{16}$$

With the combination of Euler's formula, the processing capability of definite and indefinite integrals is improved.

3.2. Examples

3.2.1. Evaluate $\int \cos^3 x dx$

From the theorem, the cube of (3) is

$$\cos^3 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 \tag{17}$$

The equation (17) can be expanded into

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$$\cos^3 x = \tfrac{1}{8}\left(e^{i3x} + 3e^{ix} + 3e^{-ix} + e^{-i3x}\right) \tag{18}$$

So:

$$\int \cos^3 x dx = \tfrac{1}{8}\int \left(e^{i3x} + 3e^{ix} + 3e^{-ix} + e^{-i3x}\right) dx \tag{19}$$

Integrate (19) term by term：

$$\int \cos^3 x dx = \frac{1}{8}\left(\int e^{i3x} dx + 3\int e^{ix} dx + 3\int e^{-ix} dx + \int e^{-i3x} dx\right)$$

$$\int \cos^3 x dx = \frac{1}{8}\left(\frac{e^{i3x}}{3i} + \frac{3e^{ix}}{i} - \frac{3e^{-ix}}{i} - \frac{e^{-i3x}}{3i}\right) + C$$

$$\int \cos^3 x dx = \tfrac{1}{8}\left(\tfrac{e^{i3x} - e^{-i3x}}{3i} + \tfrac{3\left(e^{ix} - e^{-ix}\right)}{i}\right) + C \tag{20}$$

From the theorem (4), it knows

$$\sin 3x = \frac{e^{i3x} - e^{-i3x}}{2i}$$

$$e^{i3x} - e^{-i3x} = 2i\sin 3x \tag{21}$$

Convert (20) into sine and simplify

$$\int \cos^3 x dx = \frac{1}{8}\left(\frac{2i\sin 3x}{3i} + \frac{6i\sin x}{i}\right) + C$$

$$\int \cos^3 x dx = \tfrac{1}{12}\sin 3x + \tfrac{3}{4}\sin x + C \tag{22}$$

3.2.2. Evaluate $\int_0^{2\pi} e^{ix}\sin x dx$

From the theorem, replace $\sin x$ by (4)

$$\int_0^{2\pi} e^{ix}\sin x dx = \int_0^{2\pi} e^{ix} \bullet \frac{e^{ix} - e^{-ix}}{2i} dx$$

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$$= \int_0^{2\pi} \frac{e^{i2x} - 1}{2i} dx$$

$$= \frac{1}{2i} \int_0^{2\pi} \left(e^{2ix} - 1\right) dx \tag{23}$$

Integral (23) term by term:

$$\int_0^{2\pi} e^{ix} \sin x dx = \frac{1}{2i} \left(\int_0^{2\pi} e^{i2x} dx - \int_0^{2\pi} 1 dx \right)$$

$$= \frac{1}{2i} \left(\int_0^{2\pi} \cos 2x + i \sin 2x dx - \int_0^{2\pi} 1 dx \right)$$

$$= \frac{1}{2i} (0 - \pi) = \pi i \tag{24}$$

The above integrals represent the powerful computational ability of involvement of Euler's formula. It highly simplifies the calculation steps and efficiently avoids repeated uses of trigonometric equations. Besides, in the further studies, the topics researched like Fourier series, Contour integrations and Residue theorem can be bonded together due to the high compatibility of Euler's formula.

4. Applications of the fourier series

4.1. Background

Fourier series are trigonometric infinite series that can be applied to representing and evaluating periodic functions. In 1807, for the first time, Joseph Fourier announced that any function could be expanded into trigonometric series while learning heat conduction [7]. Until 1822, he published a treatise on heat flow and introduced Fourier series [8]. He assumed there existed a function $f(x)$ defined on the interval $[-\pi, \pi]$, then $f(x)$ could be expanded into a Fourier series using Euler's formula, which was named as the complex exponential Fourier expansion [8]:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{25}$$

Where,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \tag{26}$$

4.2. Find the fourier expansion of $f(x) = x$ on the interval $[-\pi, \pi]$

Firstly, calculate the complex Fourier coefficient c_n

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$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx \tag{27}$$

Integrate (27) by parts:

$$u = x \tag{28}$$

$$du = dx \tag{29}$$

$$dv = e^{-inx}dx \tag{30}$$

$$v = \frac{e^{-inx}}{-in} \tag{31}$$

Substitute (28), (29), (30) and (31) into the partial integral formula

$$\int udv = uv - \int vdu$$

$$\int_{-\pi}^{\pi} xe^{-inx}dx = x \cdot \frac{e^{-inx}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}\,dx \tag{32}$$

Therefore,

$$c_n = \frac{1}{2\pi}\left(x \cdot \frac{e^{-inx}}{-in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}\,dx\right) \tag{33}$$

Solve (33) term by term
The first term:

$$x \cdot \frac{e^{-inx}}{-in}\Big|_{-\pi}^{\pi} = \frac{\pi e^{-in\pi}}{-in} - \frac{-\pi e^{in\pi}}{-in}$$

$$= \frac{\pi\left(e^{-in\pi}+e^{in\pi}\right)}{-in} \tag{34}$$

From equations (1) and (2), it knows

$$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n \tag{35}$$

$$e^{in\pi} = \cos n\pi + i\sin n\pi = (-1)^n \tag{36}$$

The first term (34) can be transformed into

$$x \cdot \frac{e^{-inx}}{-in}\Big|_{-\pi}^{\pi} = \frac{2\pi(-1)^n}{-in} \tag{37}$$

The second term:
Due to the orthogonality theorem in Fourier series, it knows

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$$\int_{-\pi}^{\pi} e^{-inx} dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{38}$$

From the equation (37), it knows $n \neq 0$, then

$$\int_{-\pi}^{\pi} \frac{e^{-inx}}{-in} dx = 0 \tag{39}$$

Combine (37) and (39) together

$$\int_{-\pi}^{\pi} xe^{-inx} dx = -\frac{2\pi(-1)^n}{in} \tag{40}$$

Then

$$c_n = \frac{1}{2\pi} \bullet \left(-\frac{2\pi(-1)^n}{in} \right)$$

$$= \frac{(-1)^{n+1}}{in} \tag{41}$$

Finally, the complex exponential Fourier expansion is obtained

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx} \tag{42}$$

Through the above derivation, the participation of Euler's formula provides another path to solve the Fourier expansion rather than trigonometric form. Moreover, the calculated Fourier coefficients indicate the periodicity and symmetry implicated in Fourier series. These signals exactly show the flexibility and availability of Euler's formula.

5. Conclusion

This paper has demonstrated that Euler's formula is not only a theoretical identity with the 'elegant' praise, but also a highly functional tool for figuring out problems like trigonometry and calculus via several examples. The method in the first application that changes trigonometric expressions into exponential form effectively shortens the solving process and provides a shortcut in integral manipulations, which can be proven in the following application. Furthermore, through the preliminary learning of the Fourier series, it affirms two vital natures: periodicity and symmetry, which are related to in-depth functional analysis.

Based on the highly extendable fields of both Euler's formula and the Fourier series, the future research objectives can be diverse. In applied mathematics, complex exponential functions can be used to deal with boundary value problems and eigenvalue problems that occur in fluid mechanics. Besides, Euler's formula can be used in non-cyclical Fourier transforms to process signals for practical use. What is more, jumping into physics and engineering, they may be connected to wave functions in quantum mechanics, signal modulations in electrical engineering, or even modelling in finance. The interaction between theory and practice will happen more often in the next century; it is

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more crucial to master these transformative tools for theoretical development and practical experimentation.

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