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The Role of Resultants in System of high-order equations of two variables

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Abstract. Recently, a variety of iterative methods have been developed to solve nonlinear equations and systems of nonlinear equations. These methods have been improved through Taylor interpolation polynomial integration formulas, harmonic perturbation methods, and decomposition techniques.Among them, the binary higher-order equation systems, as an important branch of nonlinear equations, is a relatively common and widely applied type of nonlinear equation, and is the research object of this paper. Through the analysis of the structure of the result formula and its application, this study aims to summarize the current methods for solving System of high-order equations of two variables, clarify the conditions for determining the existence of solutions, and emphasize the key role of the result formula in the solution process. The core significance of this research lies in integrating the theory of resultant with modern numerical methods to establish a universal solution framework applicable to various binary polynomial systems, significantly improving the efficiency and reliability of solving binary high-order equation systems in complex engineering and scientific problems, and providing theoretical support and technical references for practical applications.

Keywords: Higher-order equations, Two variables, Resultant, Determination of solutions

1. Introduction

The solution of nonlinear equations is one of the core issues in scientific computing and engineering applications. Due to the wide range of applications of nonlinear equations, there are currently several solution methods, and their development history has covered the entire process from classical algebraic methods to modern numerical analysis. Early solution methods mostly relied on algebraic transformation and symbolic computation (such as factorization, root finding), but were limited by the constraints of Abel-Ruffini theorem (there is no general algebraic solution for quintic and higher-degree linear equations), making them difficult to handle high-order or complex nonlinear equation systems. To overcome this problem, modern science proposed and analyzed methods for solving nonlinear equations, which do not require function derivatives, that is, iterative methods. Each iteration only requires the calculation of two function values [1][2]. Since the 20th century, iterative methods have gradually become the mainstream method for solving nonlinear equations due to their advantages of not requiring explicit derivative calculations and fast convergence. There are many articles on iterative methods, which do not use derivatives but require additional calculations of function values. As for the relatively simple but widely used System of high-order equations of two variables, there are several common solutions. This paper will introduce them one by one.

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Binary high-order equation systems are an important type of nonlinear equations, and their solution difficulty is much higher than that of single-variable equations. In the engineering field, binary high-order equation systems are often used to describe static equilibrium problems of complex structures (such as deflection analysis of continuous beams), kinematic constraints of mechanical systems (such as inverse kinematics solution of robot arms); in physics, they can be used to simulate conservation equations in fluid dynamics, Schrödinger equations in quantum mechanics; in mathematical research, they are closely related to the intersection problem of algebraic curves (such as the number and position of intersection points of two high-order curves); in computer science, they are widely used in collision detection in computer graphics and model parameter optimization in machine learning scenarios.

In the textbooks of advanced algebra, only the resultant elimination method for solving binary high-order equation systems is introduced, but the number of solutions is not discussed. Generally, for solving high-order equation systems with two variables, the elimination method related to the resultant is used. However, when dealing with the resultant, the calculation becomes difficult. It is necessary to develop a more convenient calculation method, specifically, by applying the principle of equivalent solutions of the equation system and the relevant knowledge of rational roots of integer coefficient equations.

2. Introduction of System of high-order equations of two variables and the resultant

A system of binary equations of higher degree refers to a pair of algebraic equations of the form $f(x,y)=0$ and $g(x,y)=0$, where f and g are bivariate polynomials with complex coefficients, and at least one of them has a degree greater than or equal to three.

In general, we can solve the high-order equation systems composed of two high-order equations with two variables by applying the resultants. However, calculating the results can occasionally be challenging. Then we can solve it by applying the principle of equivalent solutions of equation systems and the relevant knowledge of rational roots of integer coefficient equations.

Lemma: Given two non-zero polynomials over a number field P

$$f(x) = a_0x^n + a_1x^{n-1}+\ldots..+a_n \tag{1}$$

$$g(x) = b_0x^m + b_1x^{m-1}+\ldots..+b_m \tag{2}$$

If the coefficients are not all zero, then f(x) and g(x) have a non-zero common factor in P[x] if and only if there exist polynomials u(x) and v(x) in P[x] such that u(x)f(x) = v(x)g(x). Here, the degree of u(x) is less than m and the degree of v(x) is less than n.

$$\text{Let } u(x) = u_0x^{m-1} + u_1x^{m-2}+\ldots..+u_{m-1} \quad v(x) = vx^{n-1} + v_1x^{n-2}+\ldots..+v_{n-1}$$

From u(x)f(x) = v(x)g(x), it follows that the coefficients of the polynomials on both sides are equal, that is

$$\begin{cases} a_0u_0 = b_0v_0 \\ a_1u_0 + a_0u_1 = b_1v_0 + b_0v_1 \\ a_2u_0 + a_1u_1 + a_0u_2 = b_2v_0 + b_1v_1 + b_0v_2 \\ \ldots\ldots\ldots\ldots\ldots \\ a_nu_{m-2} + a_{n-1}u_{m-1} = b_mv_{n-2} + b_{m-1}v_{n-1} \\ a_nu_{m-1} = b_mv_{n-1} \end{cases} \tag{3}$$

This is a system of equations about unknowns $u_i(0 \le i \le m-1)$and $v_j(0 \le j \le n-1)$. Clearly, it is a homogeneous linear equation system with m + n unknowns and m + n equations. By the lemma,

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u(x) and v(x) must exist, that is, this equation system must have a solution. Therefore, the determinant of the coefficient matrix of this linear equation system is 0. Since row interchange and reversing the sign of an entire row do not affect the value of the determinant, the coefficient determinant matrix of this equation system can be written in the following form:

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & b_m & & \\ & b_0 & b_1 & \cdots & b_{m-1} & b_m & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & b_0 & b_1 & \cdots & b_m \end{vmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{mat

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3.3. Solving Polynomial Systems via Resultants and Linear Systems

Theorem: If (x_0, y_0) is a complex root of the system of equations $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$, then y_0 is a root of $R_x(f,g)$; conversely, if y_0 is a complex root of , then either $a_0(y) = b_0(y) = 0$ or there exists a complex number so that (x_0, y_0) is a root of the system of equations.

So, when we express f and g as polynomials of the y correlation coefficient, we can write a y-correlation equation. After we find the roots of the equation, we can substitute them back into the original equation to obtain all the complex roots of the equation [3][4].

Example: Solve a system of equations:

$$\begin{cases} y^2 - 7xy + 4x^2 + 13x - 2y - 3 = 0 \\ y^2 - 14xy + 9x^2 + 28x - 4y - 5 = 0 \end{cases} \tag{7}$$

Rewrite the equation as $\begin{cases} y^2 - (7x+2)y + (4x^2 + 13x - 3) = 0 \\ y^2 - (14x+4)y + (9x^2 + 28x - 5) = 0 \end{cases}$ (8)

Next, we construct the resultant matrix with respect to y, compute its determinant, and solve the resulting polynomial in x. This gives: x = −2, 0, 1, 2. If these values are substituted back into the system of equations, all the roots of the equation can be determined as (0, -1), (1, 2), (2, 3), (-2, 1).

3.4. Some existing solution methods

For binary systems of higher-degree equations, equivalent transformations can also be used to simplify and solve the system [5].

Lemma: The principle of equivalent solutions of the equation system [3].

First, by swapping the positions of two equations in the equation system, the resulting equation system is equivalent to the original equation system. Second, by adding (or subtracting) the same number to both sides of one equation in the equation system, the resulting equation system is equivalent to the original equation system. Third, by multiplying (or dividing) both sides of one equation in the equation system by a non-zero number and adding it to the other equation, the resulting equation system is equivalent to the original equation system.

Lemma: Let $f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0$ be an integer-coefficient polynomial. Let s and t be two distinct roots of f(x), and assume that s and t are coprime. Then:

s is a divisor of the constant term a_0 of f(x), while t is a divisor of the coefficient of the highest degree term a_m of f(x).

f(x) can be factored as

$$f(x) = (x - \frac{s}{t})q(x) \tag{9}$$

where q(x) is a polynomial with rational coefficients.

According to the principle of the third equivalent solutions, if we multiply both sides of one equation by the same non-zero polynomial and then add it to the other equation, the resulting system of equations will still be equivalent to the original system, that is:

$$f(x,y) \cdot b_i(y) - g(x,y) \cdot a_j(y) x^{j-i} = h(x,y) \tag{10}$$

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After this, we can apply the equivalent principle to reduce the degree of the equation system $\begin{cases} h(x,y) \\ g(x,y) \end{cases}$, so that the higher-degree terms in the equation system can be eliminated one by one until there is an equation that can be written in the form of x = q(y) or y = p(x). Once such a substitution is achieved, we can substitute x = q(y) into the original equation f(x, y) to determine the corresponding values of y. Then, by substituting these y-values back into g(x, y), we can solve for all the corresponding values of x.

$$\text{Example:} \begin{cases} f(x,y) = 5x^2 - 6xy + 5y^2 - 16 = 0 \\ g(x,y) = x^2 - xy + 2y^2 - x - y - 4 = 0 \end{cases} \tag{11}$$

According to the principle of identical solutions, we can obtain

$$h(x,y) = f(x,y) - 5 \times g(x,y) = -xy - 5y^2 + 5x + 5y + 4 = 0 \tag{12}$$

Thus, x can be separated and denoted as $q(y) = x = \frac{5y^2-5y-4}{5-y}$ $(y \neq 5)$. We can substituting q(y) into f(x, y) = 0.

So, we can obtain $\frac{160(y-1)^2(y+1)(y-2)}{(5-y)^2}$=0 and find four roots, y = 1, 1, -1, 2. Substituting these back in $x = (5y^2 - 5y - 4)/(5 - y)$, we can solve for x = -1, -1, 1, 2.

4. How to Determine the Number of Solutions to Binary Higher Degree Equation Systems

The determination of the number of solutions is crucial in studying various types of equations, the same for systems of binary higher-degree equations.

4.1. Methods for determining the number of solutions to System of high-order equations of two variables under normal circumstances.

Consider a system of two high-order equations $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$, where f(x) is an n-degree polynomial of x and y, and g(x) is an m-degree polynomial of x and y. We will discuss the number of solutions for this system below.

(1) Under general conditions, the system has m × n solutions. The reasoning is as follows:

On the one hand, f(x) can be decomposed into the product of n linear factors, and similarly, g(x) can be decomposed into the product of m linear factors. Thus, we obtain m×n systems of linear equations in two variables. Since this is a general case, we assume that these systems are neither all inconsistent nor all dependent. As there are m×n equations, theoretically, there should be m×n solutions.On the other hand, we apply the resultant. First, write f(x, y) as $f(x,y) = a_0(y)x^n + a_1(y)x^{n-1} + \ldots\ldots a_n(y)$; similarly, write g(x, y) as $g(x,y) = b_0(y)x^m + b_1(y)x^{m-1} + \ldots\ldots b_m(y)$. At this point, the coefficients of the equations are all related to y. Therefore, we can temporarily treat the coefficients as constants to facilitate the writing of the resultant. According to the definition of the resultant, the resultant of this system of equations can be written as:

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$$R_x(f,g)=\begin{vmatrix} a_0(y) & a_1(y) & a_2(y) & \cdots & a_n(y) & & \\ & a_0(y) & a_1(y) & \cdots & a_{n-1}(y) & a_n(y) & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & a_0(y) & a_1(y) & \cdots & a_n(y) \\ b_0(y) & b_1(y) & b_2(y) & \cdots & b_m(y) & & \\ & b_0(y) & b_1(y) & \cdots & b_{m-1}(y) & b_m(y) & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & b_0(y) & b_1(y) & \cdots & b_m(y) \end{vmatrix} \tag{13}$$

Obviously, this is a polynomial with complex coefficients in terms of y. According to the method of calculating determinants, the highest degree of y in this determinant must be m×n. Therefore, in general, this m×n-degree polynomial in terms of y must have m×n roots in the complex number field. Conversely, if x is regarded as the coefficient, the resultant about x can be obtained, and an m×n-degree polynomial about x can also be obtained, thus reaching the same conclusion.

(2) If $f(x,y)$ and $g(x,y)$have a common factor $ax+by+c$ of degree one.

At this point, since the polynomial $ax+by+c$ has infinitely many roots $x_i$andy_i, this results in there being infinitely many values of x and y that satisfy the system of equations $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$ [6].Thus the equation system has infinitely many roots. Obviously, this conclusion can be extended to higher degree common factors. That is, if $f(x,y)$ and $g(x,y)$ have a common factor, then the equation system has infinitely many roots.

(3) If $f(x,y)$ and g have no common factors, but $f(x,y)$ has at least one linear factor that is a contradiction with a linear factor of $g(x,y)$.

Thus due to the existence of the contradictory equation system, the solution should tend towards infinity. Therefore, based on condition (1), these infinite solutions should be removed at this point. Thus, the number of solutions for this equation system must be less than m×n.

In conclusion, the necessary and sufficient condition for the number of solutions of a binary system of higher-order equations to be less than the product of the degrees of the two equations is that $f(x,y)$ and g are coprime or that $f(x,y)$ has a factor that contradicts the factor of $g(x,y)$, forming a contradictory system of equations.

4.2. Some existing solution methods

For systems of binary higher-order equations, by consulting literature, it can be found that the determination of the number of solutions is closely related to the Euclidean algorithm. The following section introduces the use of the Euclidean algorithm to determine the solutions [7]:

Lemma: Two polynomials with a greatest common divisor of 0 can only be zero polynomials. The proof is as follows

According to the theorem: If d(x,y) is the greatest common divisor of f(x,y) and g(x,y), then there exist u(x,y) and v(x,y) such that d(x,y)=f(x,y)u(x,y)+g(x,y)v(x,y)

At this point, d(x,y) = 0, so f(x,y)u(x,y) + g(x,y)v(x,y) = 0. If both u(x,y) and v(x,y) are not zero, then f(x,y) and g(x,y) must be zero, so 0 is the greatest common divisor of the zero polynomial. If both f(x,y) and g(x,y) are not zero, then u(x,y) and v(x,y) must be zero, so 0 is the common divisor of u(x,y) and v(x,y), which is 0. If both f(x,y) and u(x,y) are not zero, then g(x,y) and v(x,y) must be zero, so 0 is the common divisor of g(x,y) and v(x,y), which is 0. If both v(x,y) and g(x,y) are not zero, then u(x,y) and f(x,y) must be zero, so 0 is the common divisor of u(x,y) and v(x,y), which is 0. In conclusion, 0 is the greatest common divisor of two zero polynomials.

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For the system of equations $\begin{cases} f(x,y)=0 \ f(x,y)=a_0(y)x^n+a_1(y)x^{n-1}+\ldots\ldots a_n(y) \\ g(x,y)=0 \ g(x,y)=b_0(y)x^m+b_1(y)x^{m-1}+\ldots\ldots b_m(y) \end{cases}$, we may assume that $n \geq m$.

So, according to the division algorithm with remainder $f(x,y)=Q(x,y)g(x,y)+r(x,y)$, we perform the division of f and y n times, until the remainder becomes 0 at the last step. Suppose in the last step of the division algorithm when the remainder is 0, the divisor is R, which contains the greatest common divisor off(x,y) and $g(x,y)$. Since both f and g are 0, the greatest common divisor of the two should also be 0. Let the second-to-last step of the division be $P(x,y)=q(x,y)r(x,y)+R$. At this point, the division algorithm is:

$q_1(x,y)$	$f(x,y)$ .... n times	$g(x,y)$ ....n times	$q\,(x,y)$
	$P(x,y)$ $q(x,y)r(x,y)$	$r(x,y)$ $q_1(x,y)$ R	
	R(The final divisor that includes the greatest common divisor of f and g.)	0	

(14)

For R, there are three cases: (1) It is a constant term, that is, it does not contain x or y; (2) It only contains one of x and y, that is, R is a univariate polynomial in x or y, R(x) or R(y); (3) R is a binary polynomial in x and y, R(x, y). These correspond to the three solution scenarios of this binary high-order system of equations as follows [5]:

(1) R is a non-zero constant term. At this point, the greatest common divisor of$f(x,y)$and$g(x,y)$is a constant. In this case, a non-zero constant is equal to 0. Clearly, the original equation has no solution.

(2) R is a polynomial containing only the variable y, that is $P(x,y)=q(x,y)r(x,y)+R(y)$

This way, all the solutions of the original equation $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$are transformed into the solutions of $\begin{cases} f(x,y)=0 \\ R(y)=0 \end{cases}$, achieving the goal of eliminating x. According to the equation R(y) = 0, the roots can be solved as $y_1, y_2 \ldots\ldots\ldots, y_i$, By substituting them respectively into f(x,y) = 0, the roots of x :$x_1, x_2 \ldots\ldots\ldots x_i$can be obtained. In this case, we have determined all the finite solutions of the binary high-order system of equations. Conversely, if R is a polynomial containing only x, that is, R(x) = 0, the finite solutions of x can be solved and then substituted into f(x,y) to obtain all the finite solutions of x and y.

(3) R remains a binary polynomial in terms of x and y, that is,

$$R(x,y)=h_0(y)x^n+h_1(y)x^{n-1}+\ldots\ldots h_n(y) \tag{15}$$

According to $P(x,y)=q(x,y)r(x,y)+R(x,y)$, all solutions of the original equation must be the solutions of R(x, y) = 0, which is an equation composed of a complex coefficient polynomial with two unknowns, and there are infinitely many x and y values that satisfy this equation. Therefore, at this

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time, the system of equations has infinitely many solutions, and only a finite number of increasing roots can be generated. Thus, the original system of equations has infinitely many solutions.

In conclusion, based on the Euclidean division method, we can determine the cases of no solution, finite number of solutions, and infinitely many solutions for the system of equations. And for case (2), the specific solutions can be directly obtained through the Euclidean division method to determine the number of solutions and achieve the determination of the solution

5. Conclusion

The resultant plays a crucial role in binary higher-order equation systems. It can not only be used to solve such systems but also to prove the number of solutions in general cases. We can determine whether an equation has a solution by checking if the resultant is zero. Additionally, based on the properties of determinants, it can also be used to prove the number of solutions. Its core function is to eliminate variables in the binary equation system to obtain a single-variable equation for solving. Therefore, the resultant plays a key role in determining and solving binary higher-order equation systems.

However, this research still has certain limitations. The elimination method based on the resultant may cause the degree of polynomials to increase exponentially during the calculation process, thereby causing computational complexity issues when dealing with higher-order or multi-variable equation systems. Moreover, the Euclidean algorithm proposed in this paper for determining the number of solutions still lacks an effective mechanism for handling degenerate cases, which requires further optimization in practical applications. Future research can focus on the following three aspects: Firstly, develop efficient algorithms for calculating the remainder to reduce computational complexity, such as combining sparse matrix techniques and modular arithmetic; Secondly, explore the application of the resultant theory in high-dimensional polynomial systems; Thirdly, combine symbolic computation with numerical methods to improve the efficiency of solving large-scale equation systems.

In practical engineering applications, these research results can provide theoretical support for modeling complex problems in fields such as robot path planning and celestial mechanics orbit calculations. For example, in the design of spacecraft attitude control systems, this method can be used to solve multi-body dynamics equation systems, thereby improving the accuracy of orbit prediction.

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