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Unlocking Rubik's Cube Solution Formulas with Group Theory: The Role of Commutators and Conjugates

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Abstract. This paper demystifies the Rubik's Cube solution formulas via the group theory, focusing on commutators and conjugates. First, it formalizes the Rubik's Cube as the cube group (G =<U, D, F, B, L, R>) (with (U, D, F, B, L, R) denoting clockwise 90° face rotations) and verifies that G satisfies all group axioms (closure, associativity, identity, invertibility) under rotation composition. It then extends to group actions, analyzing how G acts on the cube's facelets, edge blocks, and corner blocks to lay a theoretical basis for rotation analysis. Key findings: Commutators enable localized transformations (altering few blocks without disrupting solved parts), while conjugates realize systematic reuse of these transformations (adapting templates to any block). This shifts solving from rote memorization to "comprehensible restoration." Ultimately, the work shows group theory's role in linking abstract algebra to symmetric systems like the Rubik's Cube, improving clarity, intuition, and practical understanding.

Keywords: Group theory, Rubik's Cube, commutator, conjugate

1. Introduction

Group theory is a fundamental part of modern algebra. It looks into sets along with a binary operation that adheres to four essential axioms. Firstly, there is the axiom of closure, meaning that the results of the operation will stay within the set. Secondly, associativity is an axiom where the sequence of the operation doesn't impact the outcome. Then, there is the identity axiom which implies the existence of an element such that when it operates on others, those others remain unchanged. Finally, the invertibility axiom states that each element has an inverse which can reverse its effect. Core theorems of group theory, such as Lagrange's Theorem [1] (revealing that the order of a subgroup of a finite group must divide the order of the group, forming a key link between subgroups and group structure), Sylow Theorems [2] (characterizing the existence and number of subgroups of specific orders within finite groups, providing a powerful tool for analyzing their structure), and the Fundamental Theorem of Homomorphisms [3] (establishing a profound correspondence between group homomorphisms, normal subgroups, and quotient groups, central to the study of group relations), find application not only within mathematics—solving polynomial equation solvability and analyzing symmetries in topological spaces—but also exert a broader practical impact extending into physics (describing particle symmetries in quantum mechanics, classifying crystal structures [4]), chemistry (studying molecular symmetry [5]), computer science

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(cryptography and coding theory [6]), and numerous other fields. The significance of group theory lies in providing a unified research paradigm for abstract algebra and serving as an ideal bridge connecting abstract theory to concrete problems. Its core tools, such as commutators and conjugation, offer a rigorous analytical framework for complex transformation rules. This characteristic finds an excellent concrete manifestation in the Rubik's Cube, invented by Ernő Rubik in 1974: this 3x3x3 structure, composed of 26 smaller "cubies", has captivated countless enthusiasts with its intuitive symmetry and astonishing complexity (boasting 4.3 quintillion (4.3×10^{19}) possible configurations, with only one being the solved state).For decades, players have relied on memorizing specific move sequences (like R U R' U')—known as "heuristic algorithms" or "learned formulas"—to solve the cube. While these steps are often highly efficient, the underlying logic behind them—why a particular sequence returns cubies to their correct positions, or how seemingly disparate rotations work together to undo a scrambled state—has often remained obscure.

Consequently, this paper aims to demystifies Rubik's Cube solution formulas through group theory, focusing on commutators and conjugates. These core tools provide a rigorous framework for understanding why specific move sequences yield predictable results. By explaining how commutators create localized transformations and conjugates adapt them across the cube, this paper aims to move beyond rote memorization toward "comprehensible restoration"—where each move is understood within the cube's group structure. This paper first formalizes the cube as a group. Then, this paper dissects commutators—showing how their algebra enables targeted adjustments—and conjugates, which systematically reuse these transformations. Examples illustrate how solution formula arise naturally from these principles, translating empirical steps into mathematical logic. Ultimately, this reveals not just how but why the cube is solved, illuminating the elegant symmetry central to both puzzle and theory.

2. The structure definition of a Rubik's Cube, group theory representation, and core operation mapping—associating Rubik's Cube with group theory

A Rubik's Cube consists of three types of blocks: corner blocks (8 in total, each with three colors), edge blocks (12 in total, each with two colors), and center blocks (face centers) (6 in total, each with one color). A Rubik's Cube has 6 faces, corresponding to 6 colors; each color is divided into 9 small surfaces, which it can be called as facelets—meaning a Rubik's Cube has a total of 54 facelets. To convey information effectively, it can be assumed that fixing the viewing angle on the center block of a specific color and ensuring all center blocks remain stationary during operations. This establishes a fixed frame of reference, and through notation, a consensus on the position of each block, the orientation of each facelet, and the rotations performed on the cube can be reached.

Definition 1.1:

An "elementary rotation" is defined as rotating one face 90 degrees clockwise.

By convention, this article uses Singmaster notation to represent the elementary rotations of the 6 faces shown in Figure 1, as well as the positions of each facelet and block. Uppercase letters U (Up), D (Down), F (Front), B (Back), L (Left), and R (Right) denote the elementary rotation of the corresponding faces. Lowercase letters u, d, f, b, l, and r denote the respective faces and their corresponding center blocks. For facelets on corner blocks, the notation "xyz" is used to indicate their position, where the meaning is: the facelet located at the yz position on the x face. For example, "ufl" refers to the facelet at the fl (Front-Left) position on the u (Up) face. This notation can also be used to refer to the corresponding corner block. Similarly, for facelets on edge blocks, the notation "xy" is used to indicate their position, meaning the facelet at the y position on the x face. For example, "db" refers to the facelet at the b (Back) position on the d (Down) face. Likewise,

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this notation can be used to refer to the corresponding edge block. This notation method, which represents both facelets and blocks, does not cause confusion, as its meaning can be understood from the context [7].

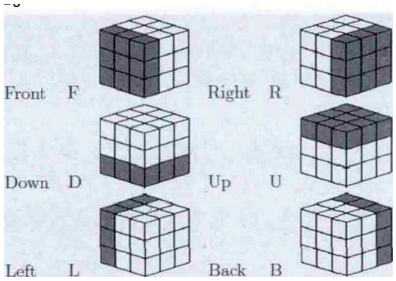

Figure 1. Six elementary rotation [8]

3. Cube group

Definition 1.2

"Rotation" can refer to a elementary rotation or any combination of elementary rotations.

Define elementary rotation composition as binary operation for any g1, g2 in {U, D, F, B, L, R} , rotation g1g2 refers to the product of the operation which means first rotating g1, then rotating g2. For example, rotation RU means first rotating the R (right) face, then rotating the U (up) face. Let C represent set of the cube state, or represent set of all facelets i.e., the legal positions of all facelets. In fact, rotation g can be considered as a function g: C→C c⟼ g(c). Thus, g1g2=g2(g1(c)) can be obtained.

Theorem 1.1

The set generated by all rotations,with composition as the operation, forms a group, called the cube group denoted as GC.

Proof:

The set of all cube rotations forms a group GC =<U,D,F,B,L,R> because:

1. Closure: Any composition of rotations (e.g.RU) is also a rotation in G.
2. Associativity: Rotation composition is associative: (gc1gc2)gc3 = gc1(gc2gc3) gcϵ GC.
3. Identity: The "no rotation" operation acts as the identity element(e).
4. Inverse: Every rotation (gc = gc1gc2.....gcn) has an inverse (gc-1 = gcn-1 .....gc2-1gc1-1) (reversing and inverting each rotation). Thus, GC satisfies all group axioms.

Additionally, it should be known that gcngcn can be expressed as gcn2 and 1GC can be expressed as gcn0. generally this multiplication does not satisfy the commutative property gc1gc2≠ gc2gc1

Definition 1.3 [9]

A group G acts on a set S when there is a map G×S→S (written (g,s) g.s) such that the following conditions hold for all sϵS

(i) Associativity: For g,hϵG and sϵS, g.(h.s) = (gh).s

(ii) Identity: The action of the identity of G on every sϵS gives s, in other words, es = s

Let C denote the set of possible states of the cube, F denote set of facelets of the cube, B denote set of blocks of the cube, EF denote set of facelets on the edge blocks of the cube VF denote the set of facelets on the corner blocks of the cube, EB denote the set of edge blocks of the cube, and VB denote the set of corner blocks of the cube.

Clearly, the following equations can be obtained:

F=EF ⋃VF EF ⋂VF= ∅

B=EB ⋃VB EB ⋂VB= ∅

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Also, if tracing any element in any of these sets, it can be found that they all be moved to an element within the set after the rotation. For instance, corner blocks can only be moved to corner blocks.

Theorem 1.2 [7]

Cube group GC can act separately on set C,F,B,EF,EB,VF,VB via g(x) where x∈any of these set.

The core purpose of group actions is to establish a mapping between 'abstract group operations' and 'concrete blocks/facelet transformations'. For any group element g (e.g., the rotation U) and set element x (e.g., the corner blocks ufl), g(x) (or g.x) denotes 'the new position/orientation of x after performing the rotation g. For instance, when the group acts on the set of corner blocks VB, U.ufl represents 'the new position of the original ufl corner blocks after executing the U rotation'. This mapping ensures that abstract group operations can directly describe the transformation rules of the physical Rubik's Cube."

4. Notation

Every Element in the cube group can be expressed by permutations of facelets(f) or blocks shown in Figure 2. The following shows the generators of the cube group represented by facelets permutation using disjoint cycle notation.(delete first two cycles of each generater to get the cycle notation of blocks permutation)

U=(ub ur uf ul)(bul rub fur luf)(bu ru fu lu)(bru rfu flu lbu)

D = (dbl dlf dfr drb)(db dl df dr)(bld lfd frd rbd)(bd ld fd rd)

F = (flu fur frd fdl)(fu fr fd fl)(ufl rfu dfr lfd)(uf rf df lf)(urf rdf dlf luf)

B = (bul bld bdr bru)(bu bl bd br)(ulb ldb drb rub)(ub lb db rb)(ubr lbu dbl rbd)

L = (luf lfd ldb lbu)(lu lf ld lb)(ufl fdl dbl bul)(ul fl dl bl)(ulb flu dlf bld)

R = (rfu rub rbd rdf)(ru rb rd rf)(urf bru drb frd)(ur br dr fr)(ubr bdr dfr fur)

The priciple works like this:Take rotation U as an example U:F→F

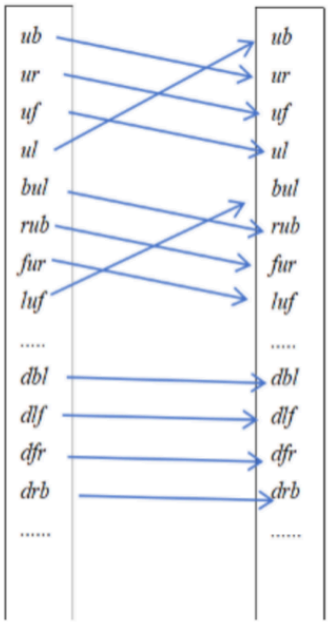

Figure 2. Diagram for better understanding of how to use cycle notations to express a permutation of facelets (picture credit: original)

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5. The formula construction for restoring the Rubik's Cube involves the principles of commutators and conjugates-------the specific manifestation of advanced group theory in the Rubik's Cube

To solve a Rubik's Cube means to take a scrambled Rubik's Cube (where corner blocks and edge blocks are misaligned in position or orientation) and adjust it, by means of legal face rotations, to the standard state—a state where all corner blocks and edge blocks are in their correct positions, and the color of each facelet matches that of the corresponding center blocks. While positioning and orienting the Rubik's Cube blocks and facelets, it is important not to disrupt the parts that have already been solved. For this reason, the solution typically relies on a Rubik's Cube formula: generally, a Rubik's Cube formula is a sequence of rotations that only alters the position or orientation of a small number of blocks, while leaving most blocks unchanged. Later the article will be introducing the key role of the commutator theorem and conjugation principle (from group theory) in constructing Rubik's Cube formula.

6. Commutator

Defintion 1.4

Let n, m be two elements. Then n-1m-1nm is a commutator of n and m denoted as [n,m].

Defintion 1.5

The support of a permutation refers to the set that contains all the elements being moved by the permutation. Denoted as sup(g) where g refers to any permutation.

Therefore, coming to Rubik's cube $sup(g) = \{f \in F \mid g.f \neq f\}\ g \epsilon\ GC$ or $sup(g) = \{b \in B \mid g.b \neq B\}\ g \epsilon\ GC$ can be obtained

Theorem 1.3 [10]

For any two permutations n and m, the support of their commutator sup(n-1m-1nm)satisfie $sup(n\text{-}1m\text{-}1nm) \subseteq Z \cup n.Z \cup m.Z$, where $Z=sup(n) \cap sup(m)$ denotes the intersection of the supports of the two permutation n an m, and g.Z represents the image of the set Z under the action of the permutation g.

From the commutator theorem, it can be concluded that for any two rotations n and m, the support (influence region) of the commutator is determined by the intersection of the support (overlapping part of the influence regions) of n and m. That is, as long as the intersection of the support of n and m is small enough, the support of n-1 m-1 n m will also become smaller. At this point, n-1 m-1 n m will only change the positions of a few blocks , while the positions of the other blocks remain unchanged, which greatly facilitates the construction of the restoration formula.

Corollary 1.1 (3-cycle theorem)

The commutator of two rotations n and m, given by $n^{-1}m^{-1}nm$, results in a minimal impact on the cube when the influence regions of n and m overlap at exactly one block, denoted as p. This commutator affects only three positions: p, n.p, and m.p, forming a cycle typically denoted as (p, m.p, n.p) in terms of directional change.

As a consequence, the formula solely permutes these three blocks while leaving all other blocks in their original positions, making it an essential tool for precise and efficient restoration steps.

Example: Supposing rotations m=ULU-1L-1 and n=R. Figure 3 and Figure 4 shows the influence region of the rotations. Clearly, it can be found that $sup(m) \cap sup(n)=\{P(fur)\}$ According to 3-cycle theorem, [n,m]=n-1m-1nm=R-1LUL-1U-1RULU-1L-1=(P m.P n.P)=(ufr ufl urb) can be obtained. Formula of changing positions of upper corner blocks can be found in Figure 5. Therefore,

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the 3-cycle formula of changing the positions of three upper corner blocks:R-1LUL-1U-1RULU-1L-1 can be constructed

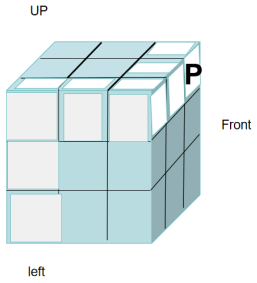

Figure 3. The influence regions of rotation m (picture credit: original)

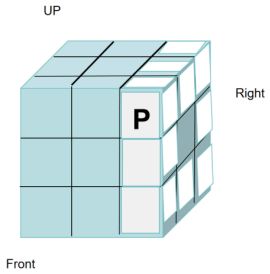

Figure 4. The influence regions of rotation n (picture credit: original)

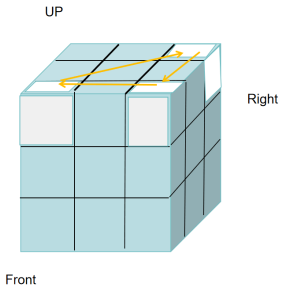

Figure 5. Formula of changing positions of upper corner blocks (picture credit: original)

7. Conjugate

Definition 1.6

In group theory, if G is a group and a, b are elements of G, then the element aba^{-1} is called the conjugate element of b by a.

Theorem 1.4

sup (aba-1) = a-1.(sup (b)) a,bϵ Group that contain permutations as element.

Proof:

Step 1: sup(aba-1) ⊆a-1.sup(b)

If aba-1(x)≠x, then a(b(a-1(x)))≠x. Applying a-1 gives b(a-1(x))≠a-1(x), so a-1(x) in sup(b). Let y = a-1(x) (where y in sup(b)), then x = a(y), so x in a-1sup(b).

Step 2:a-1.sup(b) ⊆sup(aba-1)

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If x in a-1.sup(b), let y in sup(b) satisfy x = a-1(y) so y = a(x). Since b(y)≠y, b(a(x))≠x, so a(b(a(x))≠x (i.e., aba-1(x)≠x). Thus,x in sup(aba-1).

Example

One significance of a conjugate element in restoring cube lies in generating more formulas from a given formula b, where a can be any sequence of rotations. Taking the previous 3-cycle formula as b, so the conjugate element of it is a[n,m]a-1=aR-1LUL-1U-1RULU-1L-1a-1 where m=ULU-1L-1 and n=R

According to the theorem 1.4, it can be derived that sup(a[n,m]a-1)={a-1.ufr a-1.ufl a-1.urb} with the cyclic order being (a-1.ufr a-1.ufl a-1.urb)

If setting a = R, this will generate a new 3-cycle formula of changing corner blocks (urb ufl rbd) shown in Figure 6: RR-1LUL-1U-1RULU-1L-1R-1

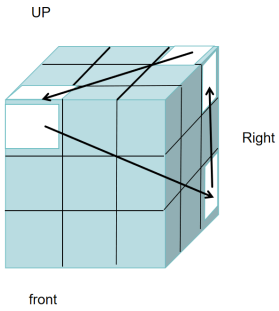

Figure 6. New formula for changing corner blocks (picture credit: original)

With the two core tools in the cube group framework—commutators (enabling localized transformations that avoid disrupting solved parts) and conjugates (realizing systematic reuse of commutator templates to target any block)—a variety of practical Rubik's Cube solution formulas can be constructed. These formulas, derived from algebraic logic rather than heuristic memorization, ultimately allow to restore a scrambled cube to its standard state.

8. Conclusion

This paper demystifies Rubik's Cube solution formulas by grounding the puzzle in group theory, with a focus on commutators and conjugates—two core tools aligned with the cube's structural properties. This paper first formalized the Rubik's Cube as a cube group G = < U, D, F, B, L, R >—a group generated by the six face rotations—verifying it satisfies all four group axioms (closure, associativity, identity, and invertibility) under the binary operation of rotation composition, as proven in Theorem 1.1. This paper then extended this framework to group actions, showing how G acts on the cube's key sets: facelets F, edge blocks E, and corner blocks VB—laying a rigorous foundation for analyzing how rotations transform the cube's state. Crucially, it can be highlighted that commutators enable localized transformations(altering only a few blocks without disrupting already solved components), while conjugates enable systematic reuse of these transformations (adapting a single commutator-based template to target any block position). This shifts cube solving from rote memorization to "comprehensible restoration," where every move is justified by the cube's group structure. Ultimately, this work underscores group theory's value as a bridge between abstract algebra and concrete symmetric systems—using the Rubik's Cube as a tangible example to demonstrate how algebraic principles explain and guide puzzle-solving.

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