Proceedings of CONF-CIAP 2026 Symposium: International Conference on Atomic Magnetometer and Applications
DOI: 10.54254/2753-8818/2026.HZ27698

Harmonic Analysis on Manifolds: Cut Loci, Laplace–Beltrami Operators, and Computational Methods

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Abstract. The author addresses the problem of solving the Poisson and Laplace–Beltrami equations on manifolds with complex topology. Classical approaches such as Fourier and spectral methods provide exact or near-exact solutions for manifolds with simple global parameterizations, but they fail in the presence of complex topologies or in the absence of highly symmetrical structure. Computational methods, while effective, are often limited by error tolerance and high computational complexity. To overcome these limitations, A distance-parameterization framework that constructs cuts on the manifold using geodesic wavefront propagation is proposed. This ensures that the resulting manifold is simply connected, without holes, allowing the use of analytic methods under well-defined boundary conditions. By using Fourier analysis, the author substituted the equation to a convolution-like form, where solutions for Minkowski metric can be easily obtained. However, it is hard to obtain an analytic solution for arbitrary metric from the equation, where the author must turn back to numerical solutions for integration.

Keywords: Poisson equation, Laplace–Beltrami operator, Manifold, Fourier transform.

1. Introduction

The study of Laplace and Poisson equations has been central to mathematics and physics since the 18th century. Originally formulated by Pierre-Simon Laplace in his investigations of celestial mechanics, the Laplace equation has since become a cornerstone in the modeling of harmonic phenomena. Fourier's introduction of Fourier series expanded its applications to heat conduction, while Bernhard Riemann and Hermann von Helmholtz further developed the analytical framework linking harmonic functions to geometry and wave phenomena. Today, the Laplace–Beltrami operator provides a natural extension of these classical ideas to curved spaces, making it indispensable in modern differential geometry, general relativity, computer graphics, and data science [1].

From a physical perspective, Laplace and Poisson equations describe equilibrium and propagation phenomena across diverse fields: potential flows in fluid dynamics, gravitational and electrostatic potentials in physics, steady-state heat conduction in engineering, and harmonic mappings in geometry processing [2,3]. Their solutions capture both local smoothness and global structural properties, making them fundamental tools for modeling continuous systems. In modern

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physics where general relativity is introduced, the will to solve Laplace and Poisson equations upon curved spacetime is even more important.

Over the past decade, a wide range of numerical and theoretical techniques have been developed to solve Laplace and Poisson equations on both flat domains and curved manifolds. Finite element methods [4], spectral discretizations [5], multigrid solvers [6], and hierarchical matrix approaches have each contributed to improving the scalability and accuracy of solutions. In geometry processing, discrete Laplace–Beltrami operators on meshes have become standard for tasks such as surface smoothing, parameterization, and shape analysis [7,8]. In applied mathematics and physics, fast solvers for elliptic PDEs continue to play a key role in simulation and optimization. Despite these advances, challenges remain in addressing manifolds of high genus, handling complex boundary conditions, and designing solvers with both high efficiency and guaranteed stability independent of target accuracy.

The goal of this work is to explore the theoretical and computational aspects of solving Poisson and Laplace equations on two-dimensional manifolds, with a particular emphasis on the interplay between metric structure, cut-locus theory, and Fourier-based methods. By combining insights from classical differential geometry with modern computational tools, the author aim to clarify the existence, uniqueness, and practical computation of harmonic solutions on surfaces of nontrivial topology.

2. Equations and basic solutions

The desire to solve Poisson/Laplacian equation for humans can be traced back to the early 18th and encounter different kinds of it in a variety of situations, from electro-magnetic waves and heat propagation to general relativity, it is encountered whenever the identification of something as a kind of wave. For a 2D manifold with metric g^{ij}, its Laplacian operator, or more precisely its Laplace-Beltrami operator has the form of

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right) \tag{1}$$

Solvers for Poisson equation can be mainly divided into theoretical solvers like Fourier transform and Spectral analysis which can manage manifolds with simple global coordinates with great isometries, while they can be transplanted to discrete scenarios. Note that the spectral method and Fourier transform greatly depend on the recursive properties of the parameterization, while for other manifolds, this recursive property may be lost or have a much more complex form. What makes this kind of method powerful is that they are precisely $O(n \log n)$, and errors are only introduced by discretization and float point number computation (discrete Fourier transforms are used for meshes).

Apart from theoretical solutions, computational solutions sometimes may not produce a real (for some oscillating solutions) continuous solution. What they have done is to approximate the solution by some error tolerance κ or number of iterations s. The complexity of these kinds of algorithms is either passively dependent on the error tolerance κ by $r(\kappa)$ where r is the rank of the matrix or actively depending on the iteration number. They typically build the discrete Laplacian matrix L by

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$$(\Delta u)(i) = \frac{1}{2A_i} \sum_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i) \tag{2}$$

where α_{ij}, β_{ij} are angles opposite edge (i, j) [9].

The Multigrid method constructs a cycle of smoothing, restricting, coarse solving, prolongate, and correcting. Using a few iterations of Jacobi, Gauss-Seidel or SOR

$$u^{k+1} = u^k + M^{-1}(f - Lu) \tag{3}$$

where M approximates L. The restriction level maps residual $r = f - Lu$ from fine grid to coarse grid with $r^c = Rr$ where R takes the average of neighbors. The coarse level solves $L^c e = r^c$ and the final prolongation interpolates coarse corrections back $u = u + Pe$ where P is usually bilinear interpolation. Note that $\sim O(n)$ is achieved on coarse operations. Multigrid method has to iterate $O(\log \kappa^{-1})$ times to reach residual reduction factor κ. With an optimal cost of $O(n)$ per iteration, the best it can reach is $O(n \log \kappa^{-1})$.

Another worth-mentioning algorithm is the Graph Laplacian method. By the cotangent formula a sparse Laplacian matrix L can be assembled and the only thing is to calculate $Lu = f$. A naive method of calculating the inverse matrix may cost $O(n^2)$ or even more [10]. Factorization or decomposition methods are used to optimize the calculation of inverse Laplacian matrix. These optimizations often depend on the rank of the Laplacian matrix and error tolerance are introduced here to reduce the rank. Typical examples are the algorithms mentioned with a complexity of

$$O\left(\left(m + n \log n (\log \log n)^2\right) \log \kappa^{-1}\right) \tag{4}$$

where κ is relatively big. While for $\kappa \to 0$, $r \to n$, the complexity reaches $\sim O(n^2)$.

3. Method proposal

3.1. Distance parameterization

To delve with complex topologies, 'cuts' are introduced. More precisely, for a manifold $\mathscr{M}$ with coordinates $\{x_i\}$ where parameter recurrence like $\forall x \in \mathscr{M} : \phi(x + T) = \phi(x)$ for some T might occur. By finding a set of points $\mathscr{C}$ and replace the manifold with $\mathscr{M}^{\text{cut}} = \overline{\mathscr{M} \setminus \mathscr{C}}$. With a nice cut, the boundary can consist of $\partial \mathscr{M}^{\text{cut}} = \partial \mathscr{M} \cup \mathscr{C} \cup \mathscr{C}$ where $\mathscr{C}$ and $\mathscr{C}$ are actually two copies of the cut $\mathscr{C}$. Assume that $\psi : \mathscr{C} \to \mathscr{C}$ is a function that maps the points who are originally the same. The usual condition is

$$F(\phi;\ \partial \mathscr{M}) = 0 \tag{5}$$

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where F is some kind of function or operator, like the Dirichlet condition means that $F=\phi\left(\partial\mathscr{M}\right)$ and the Neuman condition as $F=\left(\partial_{;n}\phi\right)\left(\partial\mathscr{M}\right)$. Now for the boundary $\partial\mathscr{M}^{\text{cut}}$, the condition

$$\phi\circ\psi\left(\mathscr{C}\right)=\phi\left(\mathscr{C}\right)\ and\ \nabla\left(\phi\circ\psi\right)\left(\mathscr{C}\right)=\nabla\phi\left(\mathscr{C}\right) \tag{6}$$

are added to the 'left' and 'right' part of the cut.

Now the author proposes that if a wave is simulated from a random point on the manifold, set of points where the wave front meets make a cut that ensures the resulting manifold is simply connected and without holes.

More precisely, let $(\mathscr{M},g)$ be a connected, compact, smooth 2-dimensional Riemannian manifold. Fix a point $p\in\mathscr{M}$ and let

$$d_p\coloneqq\int_p^x \mathrm{d}s where\ \delta\int_p^x \mathrm{d}s=0 \tag{7}$$

For each unit vector $v\in S^1\subset T_p\mathscr{M}$, let $\gamma_v:[0,\infty)\to\mathscr{M}$ be the unit-speed geodesic with $\gamma_v\left(0\right)=p$, $\dot{\gamma}_v\left(0\right)=v$. Define the cut time

$$\tau\left(v\right)\coloneqq\sup\{t>0:d_p\left(\gamma_v\left(t\right)\right)\}, \tag{8}$$

the cut locus of p is $\mathrm{Cut}\left(p\right)\coloneqq\left\{\gamma_v\left(\tau\left(v\right)\right):v\in S^1\right\}\subset\mathscr{M}$.

Theorem 1. Let $(\mathscr{M},g)$ be as above and $p\in\mathscr{M}$, then $\mathrm{Cut}\left(p\right)$ is a compact 1-dimensional CW-complex in $\mathscr{M}$ [11]. The map

$$\exp_p:\left\{tv\in T_pM:0\leq t<\tau\left(v\right),v\in S^1\right\}\to M\setminus Cut\left(p\right) \tag{9}$$

is a diffeomorphism.

Consequently, $\mathscr{M}\setminus\mathrm{Cut}\left(p\right)$ is a diffeomorphic to an open 2-disk. In particular, it is simply connected and has no holes. Then $\mathscr{M}^{\text{cut}}$ is homeomorphic to a closed disk. Note that to ensure the continuum of ϕ on $\mathscr{M}$ at $\mathscr{C}$. The condition (2) is to be enforced.

Note that now, a parameterization of s and some 'angle' can be used to describe the manifold. If the 2D manifold is represented as a 2D curved surface embedded in 3D space, mesh deformation should be used to calculate the metric manually.

3.2. Calculations

For the cut manifold and equation

$$\Delta_g\phi=-f \tag{10}$$

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with boundary conditions (1) and (2), note that it is still additive for its general solution, which is the Laplacian part. The author first considers the particular solution $\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi_p\right)=-f$. For the usual Laplace-Beltrami operator [12],

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi_p\right)=g^{ij}\partial_i\partial_j\phi_p+\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\cdot\partial_j\phi_p \tag{11}$$

Applying Fourier transforms (in unitary form), it is calculated that

$$F\left[\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\phi_p\right)\right](k)=-F[f](k) \tag{12}$$

To proceed, one can first deal with the LHS. For the LHS, it is calculated as

$$-\int \mathrm{d}q\left\{(k-q)_i(k-q)_j\mathscr{F}\left[g^{ij}\right](q)+\frac{(k-q)_j}{\mathbf{i}}\mathscr{F}\left[\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\right](q)\right\}\mathscr{F}[\phi_p](k-q) \tag{13}$$

Therefore, the following equation is obtained

$$\begin{cases} A=k_ik_jF\left[g^{ij}\right]+2ik_iF\left[\partial_jg^{ij}\right]-F\left[\partial_i\partial_jg^{ij}\right] \\ B=\frac{k_j}{\mathbf{i}}F\left[\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\right]+F\left[\partial_j\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\right] \\ K(q;k)=\{A+B\}(q) \end{cases} \tag{14}$$

Thus, the equation becomes

$$\int \mathrm{d}q\cdot K(q;k)\cdot\mathscr{F}[\phi_p](k-q)=F[f](k) \tag{15}$$

It is challenging to obtain an exact analytical solution for arbitrary metric. One must therefore resort to numerical approaches to address this problem. The detailed implementation is left to future studies. For uniform metrics like Minkowski space-time the parameter k and q is separatable in $K(q;k)$, so that a simple analytic solution can be obtained.

4. Conclusion

Ultimately, this paper investigated the theoretical and computational aspects of solving Poisson equations on 2D manifolds. The author began by reviewing the classical formulation of the Laplace–Beltrami operator and emphasized its importance in modeling equilibrium and harmonic phenomena under general metrics. The author first points out that formalized through cut locus theory, removing the cut locus from a compact surface yields a simply connected domain. This guarantees that

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harmonic solutions can be studied in a topologically simpler setting and provides an intuitive interpretation of “cutting along wavefront shocks” as a geometric tool for analysis. On the computational side, Fourier-based formulations, kernel methods, finite element discretizations, and hierarchical solvers are examined, noting both their strengths and their limitations in terms of efficiency, scalability, and dependence on target accuracy. While fast algorithms exist for Euclidean domains, extending these to curved manifolds with complex topology remains a significant challenge. Future work will focus on integrating cut locus–based reductions with high-performance numerical methods, with potential applications in geometric deep learning, mesh parameterization for computer graphics, and numerical relativity. By linking geometric insight with algorithmic development, this study contributes to closing the gap between theory and computation in harmonic analysis on manifolds.

Looking ahead, integrating the geometric insight from cut-locus theory with advanced numerical solvers appears highly promising. In this regard, future efforts should focus on developing scalable algorithms that leverage this theoretical simplification, enabling efficient computations on general curved surfaces. Such advancements could significantly benefit applications in solving differential equations with numerical approaches.

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