Proceedings of CONF-APMM 2025 Symposium: Simulation and Theory of Differential-Integral Equation in Applied Physics
DOI: 10.54254/2753-8818/2025.DL27742

What Is a Lie Algebra?

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Abstract. This paper explores Lie algebras as the infinitesimal counterparts of Lie groups, which are smooth manifolds equipped with a compatible group structure. After recalling fundamental notions of groups, manifolds, and the origins of Lie theory, we view the tangent space at the identity as a linear approximation and pose the central question: when a manifold carries a group structure, what algebraic structure does its tangent space inherit? This leads naturally to the concept of the Lie bracket. Focusing on matrix groups, we introduce linear groups and show how the commutator bracket arises from group multiplication via the exponential map. To illustrate the geometric intuition, we include figures that highlight the role of the tangent plane. We then present the formal definition of a Lie algebra and demonstrate how it encodes the local symmetries of its parent group, supplemented by illustrative examples. The paper culminates in a concise sketch of the proof of Lie's Third Theorem, which asserts that every finite-dimensional Lie algebra is associated with some Lie group, thereby completing the bridge between abstract theory and applications.

Keywords: Lie group, Lie algebra, Lie bracket, Manifold

1. Introduction

The study of symmetries has always been a central theme in mathematics and physics. From the rigid transformations of geometric shapes to the invariance properties in physical systems, understand- ing symmetry often leads to profound insights [1]. One of the most powerful mathematical structures to emerge from the study of continuous symmetries is the concept of a Lie group, introduced by Sophus Lie in the late 19th century [2]. Lie groups provide a framework to describe smooth, continuous transformations, encapsulating both the algebraic structure of groups and the geometric structure of smooth manifolds [3, 4]. In control theory, Lie algebraic methods offer a powerful framework for studying the controllability, observability, and stability of nonlinear systems by leveraging the structure of Lie groups, Lie brackets, and differential geometric principles [5]. Sophus Lie’s initial motivation stemmed from solving differential equations by exploiting their symmetry properties. In fact, Lie’s pioneering work can be seen as an extension of Galois theory to differential equations, where symmetries are represented not as discrete permutations but as smooth continuous transformations [6]. This naturally raises the question: how can one study the symmetries of a differential equation locally? The answer lies in the concept of a Lie algebra, which serves as the infinitesimal counterpart of a Lie group. Just as a smooth manifold can be approximated locally

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DOI: 10.54254/2753-8818/2025.DL27742

by its tangent space, a Lie group can be approximated locally by its Lie algebra. Concretely, for a matrix Lie group $G \subset GL_n(C)$ with Lie algebra $\mathfrak{g}$, the exponential map

$$exp : g \longrightarrow G$$

integrates each infinitesimal generator $X \in \mathfrak{g}$ into a one-parameter subgroup $\exp(tX) \subset G$. Moreover, the subgroup generated by $\exp(\mathfrak{g})$ coincides with the identity component G°. This passage from the Lie algebra to the Lie group via exponentials lies at the very core of Lie theory.

To understand this connection more generally, one observes that every smooth transformation near the identity of a Lie group induces a left-invariant vector field on the manifold. These vector fields, when closed under the Lie bracket operation, form a Lie algebra. Thus, while a Lie group describes global symmetries, its associated Lie algebra captures the local, infinitesimal symmetries.

For example, consider the general linear group $\mathrm{GL}(n, R)$, the group of all invertible $n \times n$ real matrices. Its Lie algebra $\mathfrak{gl}(n, R)$ consists of all $n \times n$ real matrices, with the Lie bracket defined as

$$[A, B] = AB - BA.$$

This algebraic structure encodes the infinitesimal transformations that generate the global group operations. In this paper, we will examine how the concept of this Lie bracket arises and its role in the broader context of Lie groups and algebras.

A remarkable converse which is stated as Lie's Third Theorem asserts that every finite-dimensional real Lie algebra arises from some Lie group. In modern form, one shows that for each finite-dimensional $\mathfrak{g}$ there exists a connected, simply connected Lie group G with the Lie algebra of G is isomorphism to

$\mathfrak{g}$. This theorem completes the bridge between local and global symmetries.

The main objective of this dissertation is to provide a self-contained introduction to these ideas and to present a complete and rigorous proof of Lie's Third Theorem. The paper is organized as follows. In Section 2, the concept of manifolds is introduced, accompanied by illustrative examples that clarify which sets qualify as manifolds and which do not. Section 3 recalls the definitions of groups and then introduce the concept of Linear group and tangent spaces. In section 4, we introduce the concept of the matrix Lie group and the abstract definitions of Lie groups. In Section 5, we define Lie algebras via tangent spaces and the Lie bracket, develop their basic properties and emphasis on both abstract and matrix examples. Finally in Section 6, we state and prove Lie's Third Theorem, showing how to integrate any finite-dimensional Lie algebra into a Lie group. Through this journey, we aim to convey both the elegant algebraic structures and the powerful geometric insights that make Lie groups and Lie algebras indispensable tools across modern mathematics and theoretical physics.

2. Manifold

To build the foundation for understanding Lie algebras, it is essential to first introduce the concept of manifolds, tangent spaces, and groups. These structures form the geometric and algebraic backbone upon which Lie theory is constructed. First, we introduce the concept of a manifold,

DOI: 10.54254/2753-8818/2025.DL27742

which plays a fundamental role in Lie theory. According to [7], we present the following three definitions.

Definition 2.1 (Locally Euclidean space of dimension n). A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of R^n. We call the pair $(U, \varphi : U \to R^n)$ a chart, U a coordinate neighborhood or a coordinate open set, and φ a coordinate map or a coordinate system on U. We say that a chart (U, φ) is centered at $p \in U$ if $\varphi(p) = 0$.

It is easy to see that a single chart cannot cover a surface of dimension 2 or higher. A locally Euclidean space is one that "locally looks like" Euclidean space. This means that around each point, we can find a small neighborhood that behaves like an open subset of R^n. For example, the surface of a sphere is globally curved, but any small patch looks flat—just like a piece of the plane. The homeomorphism φ serves as a coordinate system in that patch.

Definition 2.2 (Topological manifold). A topological manifold is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Remark 2.1. The additional conditions, Hausdorff and second countable, ensure that the space is topologically "well-behaved." The Hausdorff condition guarantees that points can be separated, and second countability ensures a manageable topology with a countable basis. These properties are essential for many analytical and geometric arguments.

Definition 2.3 (Smooth manifold). A smooth or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a differentiable structure on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a curve, a 2-dimensional manifold a surface, and an n-dimensional manifold an n-manifold.

Remark 2.2. A smooth manifold is a topological manifold equipped with additional structure that allows us to do calculus. That is, the coordinate charts not only glue together topologically, but their transition functions are smooth—infinitely differentiable. This compatibility condition allows us to define derivatives, vector fields, and eventually Lie groups and Lie algebras in a rigorous way.

Although manifolds are defined abstractly, we may introduce the following theorem which guaran- tees that we can always visualize them concretely as smooth subsets of Euclidean space. For example, a 2-dimensional torus can be embedded in R^3, and in general, we can study manifolds using the familiar geometry and analysis tools available in R^N.

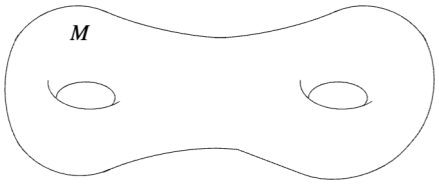

Figure 1. A smooth manifold M

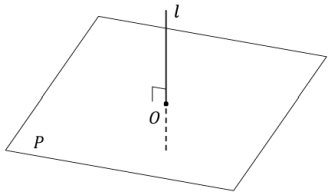

Figure 2. A straight line l intersecting the plane P or thogonally at point O

In short, a smooth manifold is a space that locally resembles R^n, satisfies nice topological conditions, and supports smooth structures compatible with calculus (see Figure 1). For Figure 2, the plane P containing the pole l, which intersects orthogonally with the plane P whose orthocenter is located at O, is clearly not a manifold, and the reason for this non-manifold structure follows directly from Definition 2.1. Since the intersection occurs at O, analyzing the neighborhood of the

DOI: 10.54254/2753-8818/2025.DL27742

point O reveals a structural problem. If O is removed, the union $P \cup l$ locally separates into two distinct components: one is connected, while the other is not. Moreover, its neighborhood decomposes into two disjoint parts—one being two-dimensional, corresponding to the plane P, and the other being one-dimensional, corresponding to the pole l. Consequently, the neighborhood of O exhibits incompatible local dimensions: one part is two-dimensional, while the other is one-dimensional. Therefore, no homeomorphism φ exists from any neighborhood of O in $P \cup l$ onto an open subset of R^n, confirming that $P \cup l$ fails to satisfy the local Euclidean property required for a manifold.

Manifolds provide the natural framework for defining smooth transformations and vector fields, which are fundamental to the theory of Lie algebras. However, the underlying geometric ideas are also intuitive. Informally, a manifold is a space that looks locally like ordinary Euclidean space. For example, the surface of a sphere or a torus is a 2-dimensional manifold because, around every point, it resembles a patch of R^2.

3. Linear group and tangent space

The following definition of a group can be found in many standard textbooks on abstract algebra.

Definition 3.1 (Group). Let G be a set with a binary operation $\cdot : G \times G \to G$. We say that $(G, \cdot)$ is a group if the following conditions are satisfied:

(1) For all $a, b \in G$, the product $a \cdot b$ is also in G;

(2) For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(3) There exists an element $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$;

(4) For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If, in addition, the binary operation is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in G$, then the group is called an Abelian group or commutative group. And in what follows, we focus on linear groups, whose structure is more accessible and intuitive.

Definition 3.2. A linear group G is any family of invertible matrices with the property that $a \in G$

implies $a^{-1} \in G$, $a, b \in G$ implies $ab \in G$.

The next definition is foundational in Lie theory and appears in many standard references on Lie algebras, including [8].

Definition 3.3 (Tangent space). Let G be a linear group. The tangent space g to G at I consists of all matrices X for which one can find a C^1 curve $a(\tau)$ that lies in G, satisfies $a(0) = I$ and $a'(0) = X$ (See Figure 3).

The concept of the tangent space can be understood as a form of linearization. When you linearize a manifold, which also serves as a group, the tangent plane acquires additional structure, namely the

Lie bracket. This process is akin to approximation theory, where taking a derivative reflects how much the structure of an object resembles its linearized approximation.

DOI: 10.54254/2753-8818/2025.DL27742

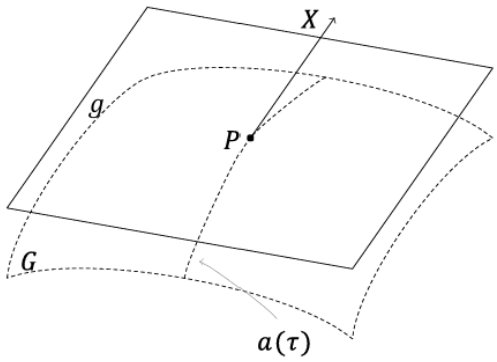

Figure 3. A smooth manifold σ with a tangent plane at point P

For definition 3.3, $a(\tau)$ is assumed to be defined for τ in some open interval around 0 in R. The C^1 condition means that $a'(\tau)$ exists and is continuous as a matrix-valued function defined on this interval. The parameter τ is real, even though the matrix space M containing G may be complex. The concept of a tangent vector arises from a fundamental observation in calculus: if $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t))$ is a differentiable path in R^k, then the velocity vector $v = \varphi'(t)$ is tangent to the path at the point $x = \varphi(0)$. More generally, a vector v is said to be tangent to a subset $S \subseteq R^k$ at a point $x \in S$ if there exists a differentiable path $\varphi(t)$, defined for sufficiently small t, lying entirely within S, such that $\varphi(0) = x$ and $\varphi'(0) = v$. For a 1-dimensional curve $\alpha(t)$, the tangent vector is simply $\alpha'(t)$. However, if we change the parameterization to $t \to 2t$, we observe that $\alpha'(2t) = 2\alpha'(t)$, showing that the scale of the tangent vector can change depending on the parameterization.

This idea also extends to higher-dimensional objects. For instance, consider a surface parameterized by two coordinates u, v. The tangent vectors to the surface can be given by the partial derivatives of the position function with respect to each parameter: $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$, where $r(u, v)$ is the position vector of a point on the surface. These vectors lie in the tangent plane at the point on the surface. Similarly, if we use a different parameterization, say x, y, the corresponding tangent vectors would be given by $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$. While the expressions for these vectors change with the choice of parameterization, the generated tangent plane is unique. This is because the tangent plane is a geometric object determined solely by the surface at a point, and it is independent of the choice of coordinates or parameterization. In contrast, the tangent plane is a geometric object: it is the plane that touches the curve or surface at a point and provides the best linear approximation of the curve or surface at that point. This leads to the notion of the tangent space, which provides a rigorous framework for describing the 'velocity' of motion through a point on a space.

In the context of Lie groups, particularly when G is viewed as a set of matrices forming a group under multiplication, we consider all smooth curves in G passing through the identity element. The set of derivatives of such curves at the identity forms a vector space, called the tangent space g at the identity. More generally, for any point $P \in G$, the tangent space at P is defined as the collection of velocity vectors arising from smooth curves in G that pass through P.

While this definition may seem abstract, it generalizes the idea of a tangent plane. At each point on a smooth surface (or higher-dimensional manifold), you can imagine a flat plane that "just touches" the surface at that point, and this is the tangent space. It contains all the directions in which one can move from p on the manifold.

DOI: 10.54254/2753-8818/2025.DL27742

Remark 3.1. From Definition 3.3, it follows that the tangent plane is a vector space of dimension k

for some $k \in N$.

A basic property of this tangent space g is:

Proposition 3.1. g is a real vector space and is closed under the bracket operation, that is, for all $X,\ Y \in g,\ \alpha,\ \beta \in R$, we have that $\alpha X + \beta Y \in g,\ [X,\ Y] \in g.$

To understand the origin of the Lie bracket [A, B] = AB − BA, we first recall a theorem from [8]:

Theorem 3.1. Let G be a linear group and $\mathfrak{g}$ its Lie algebra. Then the exponential map exp maps $\mathfrak{g}$

into G .

From this, consider $A,\ B\ \in\ \mathfrak{g}$, the Lie algebra of a Lie group G . The exponential map sends A and

B to elements $exp(A),\ exp(B)\ \in\ G$. Suppose there exists $Z\ \in\ \mathfrak{g}$ such that

$$exp(A)\ exp(B)\ =\ exp(Z).$$

The Campbell–Baker–Hausdorff formula [8] asserts that

$$Z = A + B + \tfrac{1}{2}(AB - BA) + \ldots,$$

where the higher-order terms involve iterated commutators. Thus, the bilinear operation $[A,\ B] = AB - BA$ naturally emerges as the first nontrivial correction term beyond the simple sum $A + B$ in the group multiplication law when expressed in terms of the Lie algebra.

Remark 3.2. This proposition shows that the tangent space is not only a vector space (closed under linear combinations), but also satisfies an important algebraic property: the Lie bracket of any two elements in it remains within the space. When a manifold is also a linear group, its tangent space acquires an additional structure — it becomes both a vector space and a Lie algebra. This structure turns g into a Lie algebra, which will be introduced in Section 5.

4. Matrix Lie groups and Lie groups

To provide a concrete introduction to Lie groups, we first consider matrix Lie groups, as defined in [9].

Definition 4.1 (Matrix Lie group). A matrix Lie group is a subgroup G of $GL_n(C)$ with the following property: If A_m is any sequence of matrices in G , and A_m converges to some matrix A , then either A is in G or A is not invertible.

Remark 4.1. This definition captures the idea that matrix Lie groups are “closed under smooth limits” of invertible matrices. If you take a sequence of matrices in the group that converges, the limit must either stay in the group or become non-invertible, and this ensures the group behaves nicely from an analytic perspective.

Example 4.1. The general linear groups (over R or C) are themselves matrix Lie groups. Of course, $GL_n(C)$ is a subgroup of itself. Furthermore, if A_m is a sequence of matrices in $GL_n(C)$ and A_m converges to A , then by the definition of $GL_n(C)$, either A is in $GL_n(C)$,

DOI: 10.54254/2753-8818/2025.DL27742

or A is not invertible. Moreover, $GL_n(R)$ is a subgroup of $GL_n(C)$, and if $A_m \in GL_n(R)$ and A_m converges to A, then the entries of A are real. Thus, either A is not invertible or $A \in GL_n(R)$.

Example 4.2. The special linear group (over R or C) is the group of $n \times n$ invertible matrices (with real or complex entries) having determinant one. Both of these are subgroups of $GL_n(C)$. Furthermore, if A_n is a sequence of matrices with determinant one and A_n converges to A, then A also has determinant one, because the determinant is a continuous function. Thus, $SL_n(R)$ and $SL_n(C)$ are matrix Lie groups.

While matrix Lie groups are concrete subgroups of $GL_n(C)$, a Lie group is a more abstract and general concept. It is a space that combines geometric structure (as a smooth manifold) with algebraic structure (as a group), in a way that is compatible: the group operations are required to be smooth maps with respect to the manifold structure.

Definition 4.2 (Lie group). A Lie group is a smooth manifold G which is also a group and such that the group product

$$G \times G \to G$$

and the inverse map $G \to G$ are smooth.

Remark 4.2. Every Lie group G carries both a global group structure and a local smooth structure; these interact via the requirement that multiplication and inversion be smooth maps. In particular, this compatibility endows G with a well-defined dimension (the dimension of its underlying manifold) and allows us to study its "infinitesimal" behavior by means of tangent vectors at the identity.

Having now seen both concrete (matrix) and abstract definitions of Lie groups, we are ready to explore their infinitesimal counterparts. In the next section, we will define the Lie algebra of a Lie group by examining the tangent space at the identity and equipping it with the commutator bracket. This construction lies at the heart of the correspondence between continuous symmetries and algebraic structures.

5. Lie algebras: definitions, basic properties and examples

In this section, we shift our focus from the geometric picture of Lie groups to their algebraic counterparts. Concretely, by examining the collection of tangent vectors at the identity of a matrix Lie group and endowing it with the commutator bracket, one obtains a structure known as a Lie algebra.

Definition 5.1. A finite-dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space g, together with a map $[\cdot, \cdot]$ from $g \times g$ into g, with the following properties:

(1) $[\cdot, \cdot]$ is bilinear, that is, for every $X, Y \in g, a, b \in R$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(2) $[\cdot, \cdot]$ is skew symmetric: $[X, Y] = -[Y, X]$ for all $X, Y \in g$.

DOI: 10.54254/2753-8818/2025.DL27742

(3) The Jacobi Identity holds: $[X,\,[Y,\,Z]]+[Y,\,[Z,\,X]]+[Z,\,[X,\,Y]]=0$ for all $X,\,Y,\,Z\in g$.

Remark 5.1. Roughly, the space of tangent vectors to a matrix group G at the identity is called the Lie algebra of the group. It is called an algebra because it possesses a law of composition, the bracket operation $[X,\,Y]=XY-YX$, which does not exist in the Lie group itself. This operation endows the Lie algebra with an internal structure that captures the infinitesimal behavior of the corresponding Lie group near the identity. Intuitively, a Lie algebra is a vector space with an extra operation $[\cdot,\,\cdot]$ that encodes a kind of 'infinitesimal' commutator or interaction between elements. This bracket is not associative, but the Jacobi identity serves as a weaker replacement. One classical example comes from matrices: the set of all $n\times n$ matrices with the operation $[X,\,Y]=XY-YX$ forms a Lie algebra. Lie algebras arise naturally as the tangent space at the identity of a Lie group, with the Lie bracket describing how vector fields on the group fail to commute.

The operation $[X,Y]$ is called the Lie bracket or commutator. It encodes the infinitesimal non-commutativity of transformations in the associated Lie group. However, two elements X and Y of a Lie algebra $\mathfrak{g}$ commute if $[X,\,Y]=0$. A Lie algebra $\mathfrak{g}$ is commutative if $[X,\,Y]=0$ for all $X,\,Y\in g$. We now illustrate some classical examples of Lie algebras.

Example 5.1. Let $\mathfrak{g}=M_n(R)$ and let $[\cdot,\cdot]:M_n(R)\times M_n(R)\to M_n(R)$ be given by

$$[A,\,B]=AB-BA.$$

Then $\mathfrak{g}$ is a Lie algebra.

Example 5.2. Let $\mathfrak{sl}_n(C)$ denote the space of all $X\in M_n(C)$ for which $trace(X)=0$. Then $\mathfrak{sl}_n(C)$ is a Lie algebra with bracket $[X,\,Y]=XY-YX$.

Example 5.3. Let $\mathfrak{so}_n$(R) denote the space of all real n × n skew-symmetric matrices, i.e.,
$\mathfrak{so}_n(R)=\left\{X\in M_n(R)\middle|X^T=-X\right\}$.

Then $\mathfrak{so}_n(R)$ is a Lie algebra with bracket $[X,\,Y]=XY-YX$.

In the following example, we compute the Lie algebra of $SO(n)$, the special orthogonal group, which is defined as

$SO(n)=\left\{A\in GL_n(R)\middle|A^TA=I,detA=1\right\}$.

Example 5.4 (Lie algebra of $SO(n)$). Let $SO(n)$ be the special orthogonal group, we claim that its Lie algebra is the space of all real skew-symmetric matrices:

$$so(n)=\left\{X\in Mn(R)\middle|X^T=-X\right\}.$$

Proof. Consider a smooth path $A(t)$ in $SO(n)$ such that $A(0)=I$, and define

$$X\coloneqq\frac{d}{dt}\Big|_{t=0}A(t).$$

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Since $A(t) \in SO(n)$, we have $A(t)^T A(t) = I$ for all t. Differentiating both sides with respect to t gives

$$\frac{d}{dt}\left(A(t)^T A(t)\right) = \dot{A}(t)^T A(t) + A(t)^T \dot{A}(t) = 0.$$

Evaluating at $t = 0$ and using $A(0) = I$, we obtain $X^T + X = 0$, so X is skew-symmetric. Conversely, given any skew-symmetric X, one can show that $exp(tX) \in SO(n)$ for all t, so the Lie algebra of $SO(n)$ consists precisely of the skew-symmetric matrices:

$$so(n) = \left\{ X \in M_n(R) \middle| X^T = -X. \right\}$$

This is a real vector space of dimension $k = n(n-1)/2$. The Lie bracket is given by the matrix commutator $[X, Y] = XY - YX$.

The following example examines the Lie algebra of the special linear group SL_n, we first propose a lemma.

Lemma 5.1. Let ϕ be a path in GL_n with $\phi(O) = I$ and $\phi'(O) = A$. Then $\left(\frac{\mathrm{d}}{\mathrm{dt}}(det\phi)\right)_{t=0} = \operatorname{trace} A$.

Proof. Let $\Phi(t) = (\Phi_{ij}(t))$ be the matrix-valued path in GL_n with $\Phi(0) = I$ and $\Phi'(0) = A = (A_{ij})$. By the Leibniz (complete) expansion of the determinant,

$$\det \Phi(t) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} \Phi_{i,\sigma(i)}(t).$$

Differentiating term by term and applying the product rule gives

$$\frac{d}{dt} \det \Phi(t) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \sum_{k=1}^{n} \left(\prod_{i<k} \Phi_{i,\sigma(i)}(t)\right) \Phi'_{k,\sigma(k)}(t) \left(\prod_{i>k} \Phi_{i,\sigma(i)}(t)\right).$$

Evaluating at $t = 0$, we use $\Phi(0) = I$, so $\Phi_{i,j}(0) = \delta_{ij}$. In each summand, the product $\prod_{i \neq k} \Phi_{i,\sigma(i)}(0)$ vanishes unless $\sigma(i) = i$ for all $i \neq k$. But if $\sigma(i) = i$ for all $i \neq k$, then since σ is a permutation, $\sigma(k) = k$ as well, so σ is the identity permutation. Hence the only nonzero contribution comes from $\sigma = id$, and we obtain

$$\frac{d}{dt} \det \Phi(t)\Big|_{t=0} = \sum_{k=1}^{n} \Phi'_{k,k}(0) = \sum_{k=1}^{n} A_{k,k} = \operatorname{trace}(A).$$

This completes the proof.

Example 5.6. The Lie algebra of the special linear group consists of the trace-zero matrices.

DOI: 10.54254/2753-8818/2025.DL27742

Proof. Let $\Phi(t)$ be a smooth path in SL_n with $\Phi(0) = I$ and $\Phi'(0) = A$. Since $det\,\Phi(t) \equiv 1$ for all t, we have

$$\frac{d}{dt} \det \Phi(t) = 0.$$

By the previous lemma, evaluating at t = 0, we obtain

$$\frac{d}{dt} \det \Phi(t)\big|_{t=0} = trace(A)$$

Hence $trace(A) = 0$, as required.

Remark 5.2. These examples illustrate how different structural conditions such as skew symmetry, or skew Hermitian symmetry give rise to distinct Lie algebras, each encoding the infinitesimal symmetries of its corresponding Lie group. In this sense, a Lie algebra g serves as a linearized, local approximation to the global structure of the Lie group G. The Lie bracket describes how infinitesimal transformations combine or interact, reflecting the behavior of the group operation near the identity: for matrix Lie algebras, $[X, Y] = XY - YX$ measures the failure of two transformations to commute and thus captures the curvature or noncommutativity inherent in the symmetry space. Lie algebras are particularly useful because, as vector spaces, they are easier to manipulate than the groups themselves, although the proof is nontrivial, many linear groups (including all the classical groups) are essentially determined by their Lie algebras.

6. Lie's third theorem: statement, overview and proof

In the previous sections, we have seen that the tangent space at the identity element of a Lie group encodes a rich algebraic structure—a Lie algebra. This naturally raises the question: while every Lie group gives rise to a Lie algebra, does every finite-dimensional Lie algebra come from some Lie group?

Theorem 6.1 (Lie's Third Theorem). Let $\mathfrak{g}$ be any finite-dimensional real Lie algebra. Then there exists a connected (and simply connected) Lie group $G \subset GL_n(C)$ whose Lie algebra is isomorphic to $\mathfrak{g}$.

Remark 6.1. Lie's Third Theorem completes the bridge between local (infinitesimal) symmetries and global group structure. It guarantees that the algebraic data in $\mathfrak{g}$ can always be "integrated" into a smooth manifold carrying a group law.

Several proofs of this fundamental result have been developed. The most classical approach uses Ado's Theorem, which asserts that every finite-dimensional Lie algebra admits a faithful matrix representation; by exponentiating this embedding one obtains a concrete matrix Lie group integrating $\mathfrak{g}$ (see, e.g., [10]). A more homological method, known as the Cartan's proof, constructs the group inductively via Lie algebra cohomology and central extensions, highlighting deep connections with homological algebra. Finally, a geometric construction due to Duistermaat and Kolk views the space of paths in the Lie algebra as a Banach-Lie group and then quotients by homotopy, producing the simply connected integration in a coordinate-free way (cf. [11]).

In this work, we follow the Ado-based route. While it relies on a nontrivial algebraic embedding theorem, it leads most directly to a concrete matrix group model of G and keeps the geometric

DOI: 10.54254/2753-8818/2025.DL27742

prerequisites to a minimum. In the next section we begin by recalling Ado's Theorem and the closed subgroup theorem that underlie this construction.

To succinctly state the Lie correspondence which will be repeatedly used in Section 6, we use the following notation, the Lie algebra of a linear group G will be denoted $L(G)$ rather than g when it is necessary to bring out its dependence on G. Furthermore, we shall use the characterization of $L(G)$ in terms of the matrix exponential exp: $L(G) = \{X \in M | exp\tau X \in G \ for\ all\ \tau \in R\}$. On the other hand, given a linear Lie algebra $\mathfrak{g}$, we denote by $\Gamma(g)$ the linear group generated by exp $g : \Gamma(g) = \{expX_1 \cdot expX_2 \cdots expX_k | X_1, X_2, \ldots, X_k \in g\}$. And $\Gamma(g)$ is simply called the linear group generated by g.

Theorem 6.2 (The Lie Correspondence). There is a one-to-one correspondence between connected linear groups G and the Lie algebras $\mathfrak{g}$ of the connected linear groups given by $G \leftrightarrow g$ if

$$g = L(G) \ \ or\ equivalently\, G = \Gamma(g).$$

Next, we present a proof of Lie's Third Theorem. The proof is via Ado's theorem, which asserts that every finite-dimensional real or complex Lie algebra is isomorphic to an algebra of matrices, and its outline is as follows. Before proceeding with the formal proof, we briefly outline the construction: we embed the Lie algebra into a matrix algebra by Ado's theorem, integrate to a matrix Lie group, and then pass to the simply connected cover to obtain the desired Lie group. And the following 2 Theorem are desired. Let $G \subset GL_n(R)$ be a matrix Lie group, and let $\mathfrak{g}$ be its Lie algebra. Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Then the Theorem 6.3 ensures that there exists a unique connected Lie subgroup $H \subset G$ such that $Lie(H) = \mathfrak{h}$.

Theorem 6.3. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.

The complete proof of Theorem 6.3 can be found in [9] (see Theorem 5.20). However, we outline here the main ideas involved in the proof.

We begin by constructing a candidate subgroup generated by exponentials of elements in $\mathfrak{h}$. Define

$$H_0 \coloneqq \langle exp(X) : X \in \mathfrak{h} \rangle,$$

the subgroup of G generated by exponentials of elements in $\mathfrak{h}$. Note that H_0 is a subgroup of G, but it is not necessarily closed, and therefore may not yet be a Lie group. Next, we consider the closure and connected component of H_0. Let H be the smallest closed subgroup of G that contains H_0, and denote by H° the connected component of the identity in H. By the closed subgroup theorem, H° is a Lie subgroup of G, and it is connected by definition. We then determine the Lie algebra of H°. Since H° is a connected Lie subgroup of G generated by exponentials of elements in $\mathfrak{h} \subset \mathfrak{g}$, it follows that the Lie algebra of H° is $\mathfrak{h}$. Finally, we establish uniqueness. Suppose $H' \subset G$ is another connected Lie subgroup with the Lie algebra of H' is $\mathfrak{h}$. Then the following properties hold:

(1) Both H' and H° contain $exp(X)$ for all $X \in \mathfrak{h}$.

(2) Both are connected subgroups of G with the same Lie algebra.

DOI: 10.54254/2753-8818/2025.DL27742

By the uniqueness of connected Lie subgroups corresponding to a given Lie subalgebra, it follows that $H^{'} = H^{\circ}$. This proves that there exists a unique connected Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$, completing the argument.

Theorem 6.4 (Ado's theorem, Theorem 3.17.7 in [10]). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. Then there exists a finite-dimensional vector space V and an injective Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

from $\mathfrak{g}$ into the Lie algebra of the general linear group, $\mathfrak{gl}(V)$. In other words, every finite-dimensional Lie algebra over K admits a faithful finite-dimensional linear representation.

Proof of Theorem 6.1. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Our goal is to construct a simply connected Lie group G such that the Lie algebra of G, denoted as $\mathfrak{g}_G$, satisfies $\mathfrak{g}_G \cong \mathfrak{g}$. By Ado's Theorem, there exists a injective Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(n,\ R)$ for some $n \in N$. That

is, $\mathfrak{g}$ can be identified with a matrix Lie subalgebra $\rho(\mathfrak{g}) \subset \mathfrak{gl}(n, R)$. Then by Theorem 6.3, which states that for any matrix Lie group $G \subseteq GL(n, R)$ and Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_G$, there exists a unique connected Lie subgroup $H \subseteq G$ such that the Lie algebra of H equals $\mathfrak{h}$. We apply this to the subalgebra $\rho(\mathfrak{g}) \subset \mathfrak{gl}(n, R)$ and obtain a connected matrix Lie group $H \subseteq GL(n, R)$ with H has the Lie algebra $\mathfrak{g}_H = \rho(\mathfrak{g}) \cong \mathfrak{g}$. Now, let $\widetilde{H}$ be the simply connected covering Lie group of H. Such a covering group always exists and is unique up to isomorphism. Moreover, the Lie algebra

of $\widetilde{H}$ is naturally identified with the Lie algebra of H, so the Lie algebra of $\widetilde{H}$, denoted as $\mathfrak{g}_{\widetilde{H}}$, satisfies $\mathfrak{g}_{\widetilde{H}} \cong \rho(\mathfrak{g}) \cong \mathfrak{g}$. Finally, define $G \coloneqq \widetilde{H}$, then G is a simply connected Lie group with $\mathrm{Lie}(G) \cong \mathfrak{g}$, as desired.

7. Conclusion

In this paper, we have presented a self-contained development from the concrete setting of matrix groups to the abstract framework of Lie algebras and their interplay. Beginning with the foundational concepts of manifolds, linear groups, and tangent spaces, we introduced Lie groups and examined their tangent spaces at the identity element. By endowing these tangent spaces with the commutator bracket, Lie algebras were defined as algebraic structures encoding the infinitesimal symmetries of the corresponding Lie groups, with numerous examples provided. Through classical case, we demonstrated how familiar matrix conditions give rise to rich algebraic properties within the Lie algebra framework. The centerpiece of our exposition was the proof of Lie's Third Theorem via Ado's Theorem. This construction not only demonstrates that every finite dimensional real Lie algebra arises from a connected, simply connected matrix group, but also highlights the central role of the exponential map in integrating algebraic data into global geometric structure. By closing the loop between local and global symmetries, Lie's Third Theorem cements the deep interplay between algebra and geometry that underpins modern applications in differential geometry, theoretical physics, and representation theory. Looking ahead, the technology we have developed opens several rich avenues of further study. The classification of semisimple Lie algebras, achieved through Cartan's theory of roots and Dynkin diagrams, provides a complete catalog of the building blocks

DOI: 10.54254/2753-8818/2025.DL27742

from which all complex semisimple algebras are assembled. Representation theory via universal enveloping algebras and highest weight modules reveals how Lie groups and Lie algebras act on vector spaces, with profound implications in quantum mechanics, harmonic analysis, and number theory. Finally, smooth actions of Lie groups on manifolds drive the study of homogeneous spaces, equivariant cohomology, and symmetric spaces. Each of these directions builds on the fundamental correspondence between continuous groups and their infinitesimal generators, offering powerful tools for both pure mathematics and its applications. We hope that the unified treatment presented here provides a solid foundation for readers to explore these deeper aspects of Lie theory and its many interdisciplinary connections.

Acknowledgements

I would like to thank Prof. Davi Máximo of the University of Pennsylvania for carefully reading this manuscript and providing valuable suggestions. I am also grateful to Prof. Hechun Zhang of Tsinghua University for recommending books that helped me a lot to build a foundation in Lie algebra.

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