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The Investigation of Orbit-Stabilizer Theorem and Its Applications

Yuming Li

Department of Mathematics, University of Toronto, Toronto, Canada
yuming.li@mail.utoronto.ca

Abstract. Group theory is an important area of modern mathematics, and the orbit-stabilizer theorem is a significant conclution in all of the group theory as it is the basic of many other conclusions. This article introduces theory of group action and the definition of orbit and stabilizer as a background for orbit stabilizer theorem. The theorem itself was then stated with proof. Orbit-stabilizer theorem says that when a finite group is acting on a finite set, the orbit of any element of the set must have the same size with number of left cosets of stabilizer of the element in the group. The proof was then given using the fact that two sets with a bijection between them must have same size. Examples of applications in the proof of Cauchy's theorem and Burnside's lemma were given to show the importance of orbit stabilizer theorem in Group theory. Orbit-stabilizer theorem is a central part in the proof of both of those results.

Keywords: Group theory, Orbit-stabilizer theorem, Cauchy's theorem, Burnside's lemma

1. Introduction

Group theory is a field of high significance in modern mathematics. It discusses the special structure 'group' defined with a binary operation and associativity. The idea of group theory is first developed in late 18th century to 19th century to solve problems in classical algebra, number theory, geometry, and analysis [1]. Specific groups like permutation groups and abelian groups were invented first to serve those purposes. Then general ideas of these groups were collected and gives the concrete concept of abstract group. More recent results in group theory include more divergence of group theory. Concepts such as topological groups and group presentation were invented and therefore allowed group theory to make connection with more different areas in mathematics like topology and linear algebra [1]. The definition other structures like rings and fields in algebra were also inspired by group theory. Group theory also has a lot of applications in other subjects. For example, there are algorithms in cryptography developed based on properties both abelian and infinite non-Abelian groups [2].

Among all the results of group theory, orbit-stabilizer theorem is a classic and useful conclusion. The formal version of this theorem had first appeared in "Theory of Groups of Finite Order" by Burnside in 1897 though there were already ideas appeared in former works [3]. The theorem describes the relation between the size of the orbit and the size of the stabilizer subgroup of a certain element. It helps derive many other results in group theory such as Burnside's Lemma and Cauchy's

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theorem [4]. It is also the basic idea of Pólya enumeration theorem, which gives a way to count ways of coloring certain objects with symmetries. It has varied applications in both math and chemistry for the enumeration of chemical compounds [5].

As for the importance of orbit-stabilizer theorem in group theory, this article decided to focus on discussing it. First, basic concepts of group action, orbit and stabilizer are going to be explained. Then the theorem will be stated with valid proof. Furthermore, there will be examples on both classic and recent applications of the orbit-stabilizer theorem in section two. Specifically focus on Burnsides Lemma and Cauchy's theorem. Proof is going to be included for both. The last section will be a summary of the above content and some discussion of other related topics.

2. Fundamental concepts of group action and proof of orbit-stabilizer theorem

To understand orbit-stabilizer theorem, some introduction of the concepts used in the theorem is necessary.

2.1. Fundamental concepts of group action

Group action is a method developed to describe how does the element of a group influences the element of a set by mapping them inside the set. It is one of the main parts of group theory. More details about group action and orbit-stabilizer theorem can be further found from [6, 7].

2.1.1. Definition of group action

Definition

The group action is a function from the cross-product set of a group G and a set Ω to the set, so that $.: G \times \Omega \to \Omega, .(\rho, \omega) \mapsto \rho.\omega$, with $1_G.\omega = \omega$ and $(\rho\rho').\omega = \rho.(\rho'.\omega)$.

This definition would directly define an equivalence relation.

2.1.2. Definition of equivalence classes

Definition

For $\omega, \upsilon \in \Omega$, $\omega \sim \upsilon$ if and only if $\rho.\omega = \upsilon$ for some $\rho \in G$. Then say ω and υ are in the same equivalence class.

Proof

The relation is an equivalence as:

(a) Reflexive: For $\omega \in \Omega$, there is always $1_G \in G$ such that $1_G.\omega = \omega$, implies that $\omega \sim \omega$.

(b) Symmetric: For $\omega, \upsilon \in \Omega$, suppose $\omega \sim \upsilon$, then there exists $\rho \in G$ such that $\rho.\omega = \upsilon$. Then $\omega = \rho^{-1}\rho.\omega = \rho^{-1}.(\rho.\omega) = \rho^{-1}.\upsilon$, implies that for $\rho^{-1} \in G$, $\rho^{-1}.\upsilon = \omega$, implies $\upsilon \sim \omega$.

(c) Transitive: For $\omega, \upsilon, \tau \in \Omega$, suppose $\omega \sim \upsilon, \upsilon \sim z$, then there exists $\rho, \rho' \in G$ such that $\rho.\omega = \upsilon$ and $\rho'.\upsilon = \tau$. Then for $\rho'\rho \in G$, $\rho'\rho.\omega = \rho'(\rho.\omega) = \rho'.\upsilon = \tau$, implies that $\omega \sim \tau$.

This equivalence relation defines equivalence classes that partitioned the set Ω, and it gives the definition of the orbit.

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2.1.3. Definition of orbit

Definition

Suppose group G act on set Ω, then for $\omega \in \Omega$, the orbit of ω is defined as $Orb_{G(\omega)} = \{v \in \Omega, \omega \sim v\}$.

Besides orbits, the stabilizers are defined to be subsets of the group G with elements that stabilized ω.

2.1.4. Definition of stabilizer

Definition

Suppose group G act on set Ω, then for $\omega \in \Omega$, the stabilizer of ω is defined as $Stab_G(\omega) = \{\rho \in \mathrm{G},\ \rho.\omega = \omega\}$.

This definition immediately gives that every stabilizer is a subgroup of the original group.

Lemma

$Stab_G(\omega)$ is always a subgroup of group G.

Proof

By the definition of subgroup, $Stab_G(\omega)$ is a subgroup of G if and only if $Stab_G(\omega)$ is non empty and $\forall \rho,\ \lambda \in Stab_G(\omega)$, $\rho^{-1}\lambda \in Stab_G(\omega)$. That's true as

(a) For the identity element 1_G, there is always $1_G.\omega = \omega$, implies $1_G \in G$

(b) $\forall \rho, \lambda \in Stab_G(\omega), \rho^{-1}\lambda.\omega = \rho^{-1}.(\lambda.\omega) = \rho^{-1}.\omega = \rho^{-1}.(\rho.\omega) = \rho^{-1}\rho.\omega = \omega$.

Therefore, it is able to define left cosets of $Stab_G(\omega)$ and they must partition group G, making it possible to give the orbit-stabilizer theorem.

2.2. The proof of orbit-stabilizer theorem

The orbit-stabilizer theorem indicates that the number of orbits of any element ω is the same with the number of cosets of the stabilizer of ω.

Definition

Suppose group G act on set Ω, then for $\omega \in \Omega$, $|Orb_G(\omega)| = [G : Stab_G(\omega)]$, $[G : Stab_G(\omega)]$ here represents the number of left cosets of $Stab_G(\omega)$ in G.

Proof

Define function ϕ : G/ $Stab_G(\omega) \to G\omega$, $\rho Stab_G(\omega) \mapsto \rho.\omega$ where $G/Stab_G(\omega)$ denotes the set of all left cosets. ϕ must be well defined as for any $\rho Stab_G(\omega),\ \lambda Stab_G(\omega) \in G/Stab_G(\omega)$ with $\rho Stab_G(\omega) = \lambda Stab_G(\omega)$, then there must be $\lambda^{-1}\rho Stab_G(\omega) = \lambda^{-1}\lambda Stab_G(\omega) = Stab_G(\omega)$, implies that $\lambda^{-1}\rho \in Stab_G(\omega)$. Then $\phi(\lambda Stab_G(\omega)) = \lambda.\omega = \lambda.(\lambda^{-1}\rho.\omega) = \rho.\omega$. So every coset from the domain can only be sent to one element in codomain and therefore the function is well defined.

Furthermore, ϕ is a bijection as:

(a) Injective: For $\rho Stab_G(\omega),\ \lambda Stab_G(\omega) \in G/Stab_G(\omega)$, $\phi(\rho Stab_G(\omega)) = \rho.\omega$, $\phi(\lambda Stab_G(\omega)) = \lambda.\omega$, therefore $\phi(\rho Stab_G(\omega)) = \phi(\lambda Stab_G(\omega))$ implies that $\rho.\omega = \lambda.\omega$, implies $\lambda^{-1}\rho.\omega = \lambda^{-1}\lambda.\omega = \omega$, implies $\lambda^{-1}\rho \in Stab_G(\omega)$. Then there must be $\lambda Stab_G(\omega) = \lambda\lambda^{-1}\rho Stab_G(\omega) = \rho Stab_G(\omega)$, implies that the function is injective.

(b) Surjective: For any $v \in G\omega$, there must exist $\rho \in G$ such that $\rho.\omega = v$ by definition, which would give $\phi(\rho Stab_G(\omega)) = \rho.\omega = v$, implies that ϕ is surjective.

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Then together, ϕ must be bijective, therefore the sets of all left cosets must have same cardinality with the orbit, means $|Orb_G(\omega)| = [G : Stab_G(\omega)]$.

3. Applications of orbit-stabilizer theorem

3.1. Cauchy's theorem

Cauchy's theorem is an important theorem in group theory. It confirms the existence of a prime ordered element in a group under certain conditions.

Definition

For a finite group G and a prime number p, suppose p divides $|G|$, then there must exist at least one element in G of order p[8].

Proof

Suppose that G has order n, and it acts on set $S = \left\{(a_1, \ldots a_{p-1},\ a_p) \middle| a_i \in G, a_p = (a_1 \cdot a_2 \ldots . a_{p-1})^{-1}\right\}$. Then there are n^{p-1} elements in S and the product of the elements from each tuple is always 1_G. Define $C_p =< r >$ to be a cyclic permutation group acting on S with $r.(a_1, \ldots a_{p-1}, a_p) = (a_2, \ldots a_{p-1}, a_p, a_1)$. As $a_1 = \left(a_2 \cdot a_3 \ldots a_{p-1} \cdot a_{p-1}^{-1} \ldots a_2^{-1} \cdot a_1^{-1}\right)^{-1} = (a_2 \ldots a_{p-1} \cdot a_p)^{-1}$, there must be $(a_2, \ldots a_{p-1}, a_p, a_1) \in S$. Therefore, the group action is well defined, and S is partitioned into orbits through this action.

Suppose all components of a p-tuple are the same element, then the orbit has only one element. Otherwise, the orbit must have a size of p as by the orbit-stabilizer theorem, it must divide $|C_p| = p$ and it can't be 1. Suppose there are t orbits with size 1 and k orbits with size p, then $t + kp = |S| = n^{p-1}$, implies that $t = n^{p-1} - kp$. As $p|n$, there must exist a natural number m with $t = n^{p-1} - kp = mp$. Furthermore $m \neq 0$ as $(1_G, 1_G, \ldots 1_G)$ is an example of an element with orbit size 1 so $t \neq 0$. Therefore, there must be at least p elements with orbit of size 1 and not identity. And as p is a prime, they don't have order 1, implies they have order p[4].

Cauchy's theorem is useful to confirm the existence of an order p element in a large group as it is to really find it. For example, consider the permutation group S_{11} with order $11!$, then by Cauchy's theorem, there must exist elements in S_{11} with order $2,\ 3,\ 5,\ 7$, and 11 as they are all prime numbers that divides $11!$.

3.2. Burnside's Lemma

Burnside's Lemma is another important application of orbit-stabilizer theorem. It gives a method to calculate the number of orbits created by a finite group acting on a finite set using the concept of fix points. Burnsides Lemma and its applications are discussed with more details in [9].

3.2.1. Fixed points

Definition

Let group G act on set Ω, then for any $\rho \in G$, Suppose for some $\omega \in \Omega$, $\rho.\omega = \omega$, then ω is a fixed point. The set of all elements fixed by ρ is denoted by Ω^ρ.

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3.2.2. Set of orbits

Definition

Let group G act on set Ω, then Ω/G is the sets of all orbits of this action.

3.2.3. Lemma of size of stabilizer

Definition

Suppose group G act on set Ω, then $|Stab_G(\omega)| = \frac{|G|}{|Orb_G(\omega)|}$.

Proof

Orbit stabilizer theorem indicates that $|Orb_G(\omega)| = [G : Stab_G(\omega)]$, and by definition $[G : Stab_G(\omega)] = \frac{|G|}{|Stab_G(\omega)|}$. Therefore $|Stab_G(\omega)| = \frac{|G|}{|Orb_G(\omega)|}$.

With these definitions, it allows the average number of fixed points to describe the number of orbits.

3.2.4. Burnside's Lemma

Definition

Suppose finite group G act on a finite set Ω, then $|\Omega/G| = \frac{1}{|G|}\sum_{\rho\in G}|\Omega^\rho|$

Proof

$$\sum_{\rho\in G}|\Omega^\rho| = \sum_{\rho\in G}|\{\omega\in\Omega,\ \rho.\omega=\omega\}| = |\{\rho\in G,\ \omega\in\Omega,\ \rho.\omega=\omega\}| = \sum_{\omega\in\Omega}|\{\rho\in G,\ \rho.\omega=\omega\}| = \sum_{\omega\in\Omega}|Stab_G(\omega)|.$$

Then by 3.2.3, $\sum_{\omega\in\Omega}|Stab_G(\omega)| = \sum_{\omega\in\Omega}\frac{|G|}{|Orb_G(\omega)|} = |G|\sum_{\omega\in\Omega}\frac{1}{|Orb_G(\omega)|}$. Note that the sum can be expressed as $\sum_{\omega\in\Omega}\frac{1}{|Orb_G(\omega)|} = \sum_{s\in Orb_G(\omega_1)}\frac{1}{|Orb_G(\omega_1)|}+\ldots+\sum_{s\in Orb_G(\omega_n)}\frac{1}{|Orb_G(\omega_n)|}$ with each pair of these orbits is disjoint as the orbits provides a partition of the set Ω. Furthermore, the value of each of these terms are exactly 1 as each sum is adding the inverse of the size of the orbit for the times of the size. Therefore $\sum_{\omega\in\Omega}\frac{1}{|Orb_G(\omega)|} = |\Omega/G| = \frac{1}{|G|}\sum_{\omega\in\Omega}|Stab_G(\omega)| = \frac{1}{|G|}\sum_{\rho\in G}|\Omega^\rho|$.

The application of Burnside's lemma is mostly focused on enumeration. For example, Redfield–Pólya's enumeration theorem is a direct consequence of it [10].

4. Conclusion

As a major result in group theory, orbit stabiliser theorem makes a relation between the size of orbits and the number of cosets stabilizers. It leads to more discussions about size of groups and orbits. In this article, Cauchy's theorem and Burnside's lemma are given as the examples. Cauchy's theorem specifically explains why a group with prime order must have an element of that order. Burnside's lemma says that the number of orbits is equal to the average number of fixed points. Burnside's lemma have a lot of examples of applications in enumeration, which occurred in many subjects. In future studies, it will be useful to review the applications of Burnside's Lemma, such as Redfield–Pólya's enumeration theorem, and to explain their significance.

References

[1] Kleiner, I. (1986). The evolution of group theory: A brief survey. Mathematics Magazine, 59(4), 195-215.

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[2] Shpilrain, V., & Zapata, G. (2006). Combinatorial group theory and public key cryptography. Applicable Algebra in Engineering, Communication and Computing, 17(3), 291-302.
[3] Burnside, W. (1909). Theory of groups of finite order. Messenger of Mathematics, 23, 112.
[4] McKay, J. H. (1959). Another proof of Cauchy's group theorem. Amer. Math. Monthly, 66(2), 119.
[5] Benassi, E. (2013). Application of Redfield–Pólya's Theorem to the enumeration of the substitution isomers of linear polycyclic aromatic hydrocarbons. Journal of Mathematical Chemistry, 51(9), 2264-2270.
[6] Rose, J. S. (1978). A course on group theory. CUP Archive.
[7] Pyone, A. (2018). Orbit-Stabilizer Theorem And Consequences. J. Myanmar Acad. Arts Sci. 2018 Vol. XVI. No.3
[8] Yang, X., & He, Y. (2022). A proof of Cauchy theorem by action of group. In 2nd International Conference on Applied Mathematics, Modelling, and Intelligent Computing (CAMMIC 2022) (Vol. 12259, pp. 369-373). SPIE.
[9] Jin, J. (2018). Analysis and applications of burnside's lemma.
[10] Harary, F., & Palmer, E. (1967). The enumeration methods of Redfield. American Journal of Mathematics, 89(2), 373-384.