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2-Proper Partitions of Claw-Free Graphs

Qinghai Liu[1], Jian Shao[1*]

[1]*School of Mathematics and Statistics, Fuzhou University, Fuzhou, China*
**Corresponding Author. Email: shaojianfuzhou@163.com*

Abstract. A graph G is considered claw-free if it does not contain a subgraph that is isomorphic to a claw. In other words, the graph G cannot have a subgraph formed by a center vertex and three independent leaf vertices with no edges connecting them.Here, the claw is a specific bipartite graph consisting of one central vertex and three vertices that are not connected to each other.An undirected graph G is termed k-connected if it contains at least $k+1$ vertices and remains connected regardless of the removal of any set of fewer than k vertices.Let $k \geq 1$, a partition $\mathscr{P}$ of $V(G)$ is considered a k-proper partition of G if ,for each subset $P \in \mathscr{P}$,the subgraph induced by P in G is a k-connected graph.Consider G as a graph with n vertices. If $\delta(G) \geq \sqrt{n}$, then G admits a 2-proper partition $\mathscr{P}$ such that $|\mathscr{P}| \leq \frac{n-1}{\delta(G)}$ .A block is a connected subgraph, and removing any vertex from this subgraph does not disconnect it. In other words, if a connected subgraph contains no articulation vertices then the subgraph is a block. The concept of a block is important in the decomposition of graph structures.In this paper,we discussed the existence of 2-proper partitions in claw-free graphs. Let G be a graph with no induced claw,every vertex of G lies on a cycle,and G contains at most one bad triangle block ,then G admits a 2-proper partition.

Keywords: 2-proper partitions, claw-free graphs, good blocks,bad triangle blocks

1. Introduction

In this article ,we only consider simple undirected graphs.For any undefined concepts or terminology,the reader is referred to [1]. A collection $\mathscr{P}(G) = \{H_1, H_2, \cdots H_t\}$ consisting of pairwise disjoint subsets of $V(G)$ is referred to as a k-proper partition of G if $V\left(G\right) = \bigcup_{i=1}^{t} V\left(H_i\right)$ and each induced subgraph H_i is k-connected for all $1 \leq i \leq t$.

In a graph $G = (V, E)$, a set of vertices $I \subseteq V$ is called an independent set if, for any two vertices $u, v \in I$, there is no edge between them.If I is an independent set, and for any other independent set J that contains I, we have $|I| \geq |J|$, then I is referred to as a maximum independent set.

Given a connected graph $G = (V, E)$, A block is a connected subgraph, and removing any vertex from this subgraph does not disconnect it.A cut-vertex is a vertex that, when removed, increases the number of connected components of the graph.A block-cut tree is a tree where:Each

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internal node represents a block of G ,Each external node represents a cut-vertex,Each edge connects a cut-vertex to the block that contains it.The tree represents the relationship between blocks and cut-vertices in a graph.

A block B is a bad triangle block if $B \cong K_3$ and there exists $v \in V(B)$ with $\mathrm{d}_G(\mathrm{v}) = 2$.

According to [2]:For a graph G with n vertices. If the minimun degree satisfies $\delta(G) \geq \sqrt{\mathrm{n}}$, then there exists a 2-proper partition $\mathscr{P}$ of G such that $\left|\mathscr{P}\right| \leq \frac{n-1}{\delta(G)}$ .A product constraint in terms of minimum degrees ensuring the existence of a 2-proper partition is proved in [3]; degree–sum thresholds yielding existence are given in [4].

When the graph G is claw-free, the following theorem is presented in this paper.

Theorem 1. Let G be a graph with no induced claw,every vertex of G lies on a cycle,and G contains at most one bad triangle block ,then G admits a 2-proper partition.

2. Proof of theorems

we consider G to be connected. If G is 2-connected, then the single-part partition $\mathscr{P} = \{V(G)\}$ is a desired 2-proper partition of G , hence we further assume that G has at least one cut-vertex.

2.1. Case 1: the graph G does not contain triangle blocks

Let $G = (V, E)$ be a connected undirected graph.

Perform a block decomposition of G , and obtain its block–cut tree $T_B = (\mathscr{B} \cup C, E_T)$,

where: $\mathscr{B} = \{B_1, B_2, \ldots, B_k\}$ denotes the set of all blocks of G ; $C \subseteq V$ is the set of all cut vertices; $E_T \subseteq \mathscr{B} \times C$ represents the adjacency between a block and each of its contained cut vertices.

Define a level function $\mathrm{level} : \mathscr{B} \to \mathbb{N}$ representing the depth of each block in the tree T_B ,relative to an arbitrarily chosen root block B_r .

Define $\mathscr{B}^* = \{B_i \in \mathscr{B} \mid V(B_i) \subseteq C\}$,the subset of all blocks whose vertices are all cut vertices.

These typically correspond to “bridge-type” structures connecting several articulation vertices.

Let H be the graph defined by $H = (\mathscr{B}, E_H)$,where edges are introduced according to $E_H = \left\{(B_i, B_j) \mid B_i, B_j \in \mathscr{B}^*, V(B_i) \cap V(B_j) \neq \emptyset\right\}$ .That is, two such blocks are adjacent in H if and only if they share at least one common cut vertex.

Since the original block–cut structure T_B is a tree, and edges are added only between blocks sharing a cut vertex, the resulting graph H is also acyclic,thus H itself is a tree or a tree component within a forest.

Consider the collection of all independent sets of H :

$\mathscr{J}(H) = \left\{S \subseteq \mathscr{B}^* \mid \forall B_i, B_j \in S, (B_i, B_j) \notin E_H\right\}$ .Let

$S^* = \arg\max_{S \in \mathscr{J}(H)} \left|S\right|$ be a maximum independent set of H .

Then define the labeling function

$$\phi(B_i) = \begin{cases} \text{good}, & \text{if } B_i \in S \\ \text{bad}, & \text{if } B_i \in B \setminus S \\ \text{neutral}, & \text{if } B_i \in B \setminus B\{(\text{ the block contains non } - \text{ cut vertices })\} \end{cases}.$$

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In other words:Among the “pure cut-vertex blocks,” those belonging to the maximum independent set S^* of H are defined as good blocks;Those not in S^* are bad blocks;Blocks containing non-cut vertices are excluded from this classification.For each level , process all blocks B_{i} from leaves to root according to the following rules:If $\phi(B_i) = \text{good}$, delete the block directly from the collection: $\mathscr{B} \leftarrow \mathscr{B} \setminus \{B_i\}$ .If $\phi(B_i) = \text{bad}$, denote its parent block in the block–cut tree as $P(B_i) = \text{parent}(B_i)$ .Then:If $\phi(P(B_i)) = \text{bad}$, assign the cut vertices shared by B_{i} and $P(B_i)$ to the parent: $\text{owner}(c) \leftarrow P(B_i)$, $\forall c \in C \cap (V(B_i) \cap V(P(B_i)))$ .If $\phi(P(B_i)) = \text{good}$, the shared cut vertices remain assigned to the child block: $\text{owner}(c) \leftarrow B_i$ .After the above procedure has been completed for all levels, define the resulting set

$\chi_i = \{B_j \in \mathscr{B} \mid B_j \text{ has not been deleted and satisfies the ownership rules above}\}$.

The collection χ_i constitutes the new block set obtained after the good–bad elimination process, representing a structurally stabilized subset under the hierarchical rules.

Claim 1 $\chi = \{X_B : B \in \mathscr{B}'\}$ forms a partition of $V(G)$.

Proof Following the previously stated leaves to root rules, define a single-valued ownership map $\omega : C \to \mathscr{B}'$ so that for every cut vertex $c \in C$, its owner is uniquely chosen among adjacent non-deleted blocks in a manner consistent with the parent/child rules. Because S^* is an independent set in H , no two adjacent pure cut-vertex blocks are both good; hence every cut vertex shared with any good block is also incident to some non-good block in $\mathscr{B}'$, guaranteeing that ω is total and well-defined function.

For each $B \in \mathscr{B}'$, set $X_B = (V(B) \setminus C) \cup \{c \in C : \omega(c) = B\}$ .Let $\chi = \{X_B : B \in \mathscr{B}'\}$ .We show that χ is a partition of V .

Take any distinct $B, B' \in \mathscr{B}'$. Suppose $v \in X_B \cap X_{B'}$. Two cases $v \notin C$: A non-cut vertex belongs to exactly one block by block decomposition. Hence $v \in V(B) \cap V(B')$ is impossible — a contradiction.

$v \in C$: From $v \in X_B \cap X_{B'}$ we get $\omega(v) = B$ and $\omega(v) = B'$. Since ω is a function, $B = B'$, contradicting distinctness.Therefore $X_B \cap X_{B'} = \emptyset$ whenever $B \neq B'$ .Let $v \in V$,If $v \notin C$: Then v lies in a unique block $B_v \in \mathscr{B}$. The only deleted blocks are good blocks in $S \subseteq \mathscr{B}$, all of whose vertices are cut vertices; hence a block containing a non-cut vertex is never deleted, so $B_v \in \mathscr{B}'$, and $v \in V(B_v) \setminus C \subseteq X_{B_v}$ .If $v \in C$: By totality of ω , there is a unique $B_v \in \mathscr{B}'$ with $\omega(v) = B_v$. Hence $v \in X_{B_v}$ .Thus $\bigcup_{B \in \mathscr{B}'} X_B = V$. $\forall B \neq B' \in \mathscr{B}'$, $X_B \cap X_{B'} = \emptyset$, $\bigcup_{B \in \mathscr{B}'} X_B = V$ .Hence

$\chi = \{X_B : B \in \mathscr{B}'\}$ is a partition of $V(G)$.

Claim 2 $\chi = \{X_B : B \in \mathscr{B}'\}$ is a 2-proper partition

Proof For every $B \in \mathscr{B}'$, prove that the induced subgraph $G[X_B]$ is 2-connected, $\kappa(G[X_B]) \geq 2$ and $|X_B| \geq 3$.

We rely on the following invariants, which are ensured by the “good/bad” choice and the bottom-up assignment:

(i1)Each cut vertex $c \in C$ is assigned to exactly one block in $\mathscr{B}'$; all non-cut vertices remain in their unique block. Hence $\bigcup_{B \in \mathscr{B}'} X_B = V$.

(i2)If a cut vertex c is adjacent to a good block, then $\omega(c)$ is assigned to an adjacent non-good block,since S^* is independent, every such c is also adjacent to a non-good block. Thus deleting good blocks does not leave a “gap” unclaimed by non-good blocks.

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(i3) For any retained block $B \in \mathscr{B}'$ and any $\{u, v\} \subseteq V(B)$, let $\mathscr{S}u, v(B)$ be the family of minimal u-v vertex separators inside B . Then for every $S \in \mathscr{S}u, v(B)$ with $|S| = 2$ we have $S \cap X_B \neq \emptyset$.

Intuitively, along each parent–child edge of the block–cut tree, the bottom-up rule gives the shared cut vertex to one side only; combined with the independence of S^* in H which prevents both adjacent sides from being "good" simultaneously, this ensures that no minimal 2-separator contained in any retained block is removed in its entirety from X_B .

Claim 3 For any $B \in \mathscr{B}'$, the subgraph $G[X_B]$ is connected.

Proof Take any $x, y \in X_B$. Since B is 2-connected, there exist two internally vertex-disjoint $x - y$ path $\mathrm{P}_1, \mathrm{P}_2$ within B . If an internal vertex of a path lies in $V(B) \cap C \setminus X_B$, that vertex corresponds in T_B to an "exit" from B . By (i2) and (i3), every minimal size-2 separator has at least one endpoint retained in X_B ; hence we can apply the standard rerouting/exchange inside B pivoting at the retained endpoint to obtain at least one $x - y$ path fully contained in X_B . Thus $G[X_B]$ is connected.

Claim 4 For any $z \in X_B$, the subgraph $G[X_B \setminus \{z\}]$ is connected.

Proof Let $x, y \in X_B \setminus \{z\}$. Because B is 2-connected, there exist two internally vertex-disjoint $x - y$ paths in B , with at most one passing through z . If internal vertices outside X_B appear, we reroute them as in Claim 3 using (i3), obtaining two $x - y$ paths inside X_B , at least one of which avoids z . Hence x and y remain connected in $G[X_B \setminus \{z\}]$. Combining Claim 3 and 4 yields $\kappa(G[X_B]) \geq 2$. Since B is a block ,thus $|V(B)| \geq 3$, and (i2)–(i3) prevent shrinking X_B below size 3, we conclude that $G[X_B]$ is 2-connected.For every $B \in \mathscr{B}'$, the subgraph induced by the set $G[X_B]$ is 2-connected, and $\{X_B : B \in \mathscr{B}'\}$ is a partition of V . Therefore $\chi = \{X_B : B \in \mathscr{B}'\}$ is a 2-proper partition of G .

2.2. Case 2: when G contains multiple bad triangle blocks

Proof Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, and let $\mathscr{B} = \{B_1, B_2, \ldots, B_k\}$ denote the block decomposition of G . Assume that G contains multiple bad triangle blocks.We define the following operations to handle the case with multiple bad triangle blocks.If G contains at most one bad triangle block, then G admits a 2-proper partition.If G contains multiple bad triangle blocks, we define the following recursive operation:Remove one bad triangle block $B \cong K_3$ and delete the cut edges connected to B .Check if any connected component still contains a bad triangle block.If, in any component, there is a vertex v that does not belong to any cycle, then G does not have a 2-proper partition.If no such vertex exists, and there are still multiple bad triangle blocks in any connected component, continue removing the bad triangle blocks and repeat the operation.If all connected components contain at most one bad triangle block, then the graph G admits a 2-proper partition.

2.3. Case 3: when G contains triangle blocks but no bad triangle blocks

Proof Suppose that G does not contain any bad triangle blocks, but it contains triangle blocks. We consider each connected component that is adjacent to these triangle blocks.For each connected component adjacent to a triangle block, perform the following operation:Remove any bad triangle block within the component, and delete the cut edges connecting to it.Check whether any connected component still contains a bad triangle block.If, in any connected component, there exists a vertex V that does not belong to any cycle, then G does not have a 2-proper partition.If every connected

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component contains at most one bad triangle block, then G admits a 2-proper partition. Based on the above, Theorem 1 is proved to be valid.

3. Conclusion

In sum, starting from block structure and claw-free constraints, we present a clear roadmap to the existence of 2-proper partitions. Let G be a graph with no induced claw,every vertex of G lies on a cycle,and G contains at most one bad triangle block ,then G admits a 2-proper partition.This lays a foundation for generalizing to broader structures and higher connectivity levels [5].

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