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Discussion on the Existence of Always an Interior Square Inside Any Simple Closed Curve

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Abstract. This paper addresses the "interior square problem" posed by Toeplitz in 1911: whether every simple closed curve contains a square's four vertices. While proven for special curves (smooth, convex, etc.), it remains open for general ones. While proven for special curves (smooth, convex, etc.), it remains open for general ones. We propose a staged framework.For convex curves, via symmetric tangent midpoint trajectories, a rotation parameter, and the mediator theorem, we rigorously prove inscribed squares exist. For general curves, we use point-pair space topology, mapping to 3D space (midpoints as horizontal, segment lengths as vertical coordinates). Topological analysis shows the resulting space is a Möbius band; its non-embeddability in 3D implies unavoidable self-intersections, corresponding to critical configurations (including squares).By integrating geometric and topological methods, constructive proofs for special cases are given, and topological evidence for the general conjecture is provided, which advances understanding of curve symmetries and geometric-topological interactions.

Keywords: simple closed curves, midpoint trajectory construction, mediator theorem, topological embedding, Möbius bands, self-intersection phenomenon, point-to-space

1. Introduction

In plane geometry, a classical question is whether for any simple closed curve Ω, it is always possible to find a quadrilateral whose four vertices fall exactly on the curve Ω and form a square. This problem originates from a conjecture formulated by Toeplitz at the beginning of the 20th century, and although it has been proved in some special cases (e.g., smooth curves, convex curves, segmented polygons, etc.), it is still an open problem in the most general case. In this paper, an illuminating construction method is used to justify it.

The problem has attracted the attention of many mathematicians over the past century or so. Various geometric constructions or topological methods have successfully proved the existence of squares for smooth curves, convex curves, polygons, and even certain segmented continuous curves. For example, the work of Stromquist, Tao, and Makeev et al. has made important progress under different assumptions. However, in the general case of simple closed curves (Jordan curves) in the universal sense after removing the constraints of smoothness, convexity, continuous differentiability,

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etc., a comprehensive solution to the problem is not yet available.Benjamin Matschke systematically sorted out the research results of this problem in different scenarios and pointed out that the general case without additional constraints is still a major unsolved problem in geometric analysis [1].

The reason is that this problem is characterised by both the local nature of the geometric construction and the holistic nature of the topological structure: the four vertices of the square have to satisfy strict angle and distance constraints, and the distribution of their positions on the closed curve is strongly influenced by the shape of the curve as a whole, which makes it difficult to apply the traditional algebraic methods and differentiation techniques directly.

For this reason, the strategy of sub-case construction-general extension is adopted to carry out the study. Starting from two special cases, the nature of the problem is analysed step by step and modeled structurally.

• The first case is a convex closed curve. In this context, we design a geometric construction method based on the control of midpoint trajectories and rotation parameters, and successfully prove the existence of an interior square by means of the median theorem

• The second class of cases considers arbitrary general closed curves, i.e., the case where the convexity and regularity conditions are removed. In this context, we turn to the construction of the interior rectangle, and by mapping 2D point pairs to line structures in 3D space, we further reveal the inevitability of self-intersection in this mapping system by means of the theory of topological spatial non-embeddability, thus providing indirect evidence for the existence of squares with "a certain kind of stable symmetric structure". indirect evidence for the "existence of some stable symmetric structure [2] " of squares.

These constructions not only retain the intuition of geometric derivation, but also introduce modern mathematical tools such as topological homology, the immovable point principle, and parameter space analysis to build a framework for the study of the integration of geometry and topology. These constructions not only retain the intuition of geometric derivation... Our study shows that even if it is not possible to directly construct a square on any simple closed curve, a strong indirect support for the conjecture can be provided by observing the structural conflicts and symmetries inevitably arising from its mapping system.

The paper is structured as follows: Section 2 introduces the geometric construction and angle control in the case of convex curves; Section 3 explores the three-dimensional mapping and topology under general closed curves; and Section 4 summarises our main findings and Section 4 summarises the main findings and proposes directions for future research, including the prospects for the application of high-dimensional generalisation and algebraic topological methods. including the prospects for the application of high-dimensional generalisation and algebraic topological methods.

2. Method construction and theoretical analysis

2.1. Convex curves: inscribed squares via midpoint & rotation

Step 1: Construct the midpoint trajectory curve.

Let Ω be a simple closed curve. We choose a line $L1$ whiistangent to Ω at a certain point. Find a line $L2$ tangent to Ω on the other side with the same slope. Let , $L1L2$ intersect Ω at , $A1A2$ and , $B1B2$ and take the midpoints 2 $M1 = A1 + A2$, 2 $M2 = B1 + B2$.

By connecting all possible $M1$ (corresponding to different locations of $L1$), we get a continuous curve $\gamma 1$; the same method is used to make another set of midpoint trajectory curve $\gamma 2$ for $L3$.

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Step 2: Find intersection points and construct parallel lines

Let $\gamma 1$ and $\gamma 2$ intersect at point O, and make two straight lines parallel to $L1$ and $L3$, which intersect with Ω at A,B and C,D. According to the construction, AO=BO, CO=CO:

- AO=BO, CO=DO, i.e., AB and CD are segments with OO as their midpoint;

- AB ∥ , CD ∥ $L1$, and since $L1 \perp L3$, AB⊥ CD.

As a result, the quadrilateral ABCD is a rhombus and AB⊥ CD and hence a sufficient necessary condition for it to be a square is that one of the angles is a right angle.

Step 3: Application of the Parameters of Rotation and the Median Theorem [3]

A continuous rotation of the tangent line is considered to control the shape of the quadrilateral ABCD. It is noticed that the angles of the quadrilateral, ∠ACB, change continuously during the rotation goes from $< 90^\circ < 90^\circ$ to $> 90^\circ > 90^\circ$, there must be some angle of rotation such that $\angle ACB = 90^\circ$ $\angle ACB = 90^\circ$, where ABCD is a square, according to the Median Theorem.

Although this method is intuitively effective under convex curves, the key lies in the angle change induced by continuous mapping and the median value theorem. If the continuity of the midpoint trajectory curves can be guaranteed, the method is also applicable to a wider range of curves (e.g., smooth curves, Jordan curves, etc.). If more topological tools (such as homology or algebraic topology) can be introduced, it may be possible to generalise to the more general case of curves.

Definition 1(Midpoint Trajectory Curve):Let Ω be a convex simple closed curve in the plane. We consider a family of lines L which are tangent to Ω at two locations: A_1, A_2 and B_1, B_2 on each side of the curve. Denote the midpoint of each pair of tangent points as:

$M_1 = (A_1 + A_2)/2, M_2 = (B_1 + B_2)/2$

As the line L rotates (changes angle), M_1 and M_2 form a continuous trajectory in the plane, denoted γ_1 and $\gamma 2$, respectively, which we call the "midpoint trajectory curve".

Definition 2(Inside Square):Let Ω be a simple closed curve, if there are four points A, B, C, D are located on Ω, and the quadrilateral ABCD meets the following conditions, then it is called an interior square:

1.The four sides are equal in length: AB = BC = CD = DA

2.All four angles are right angles: $\angle ABC = \angle BCD = \angle CDA = \angle DAB = \angle DAB = 90^\circ$

Theorem 1 (Construct a square based on the midpoint trajectory):If the midpoint trajectories γ_1 and $\gamma 2$ intersect at point O, then two lines parallel to the original tangent line can be made through O to intersect the curves Ω at points A, B, and C, D. In this case, the quadrilateral ABCD satisfies the following geometry.

The quadrilateral ABCD satisfies the following geometric relationship:

-AO = BO, CO = DO, i.e., AB and CD are both segments with O as their midpoints

-AO = BO, CO = DO, i.e. AB is parallel to the original tangent L_3 and CD is parallel to $L1$ and $L1$ is perpendicular to L_3, so AB ⊥ CD.

Therefore, ABCD is a rhombus; if one of its angles is a right angle, then ABCD is a square.

Theorem 2(Median Theorem):If a function f is continuous on the interval [a, b] and f(a) and f(b) have opposite signs, i.e., f(a) < 0 and f(b) > 0 or vice versa, then there must exist some c ∈ (a, b) such that f(c) = 0.“The geometric relationship between midpoint trajectories, tangent lines, and the constructed quadrilateral (as required for proving the existence of inscribed squares) is visually illustrated in Figure. 1.”

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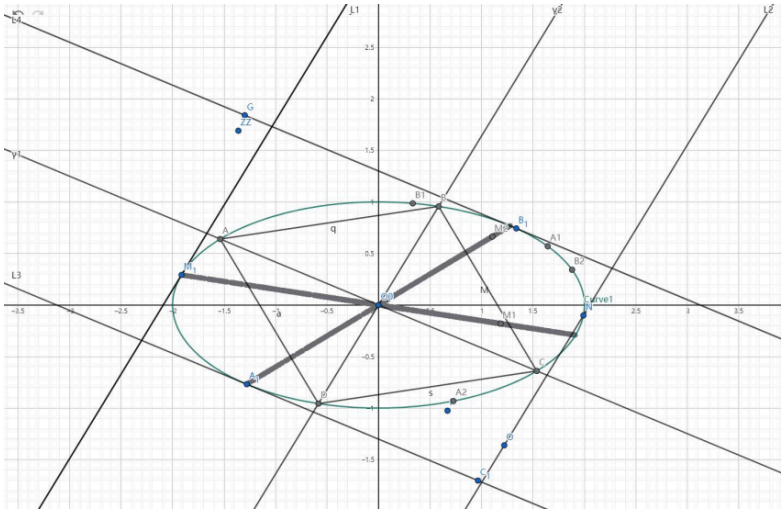


Figure 1. Midpoint trajectory & inscribed square for convex curves

2.2. General curves: interior rectangles, convex generalization

The problem is now approached from another perspective - that of topology and mapping spaces - and the spatial structure associated with the "square-interconnection" problem is shown. The question we are concerned with is: given that a is a simple closed curve in the plane (i.e., a closed curve that is not self-intersecting), is it possible to naturally map all pairs of points (a,b) (a,b $\in$ a,b $\in$ a) to line segments in three-dimensional space such that these segments do not intersect each other?

In this paper, the unavoidability of this self-intersection phenomenon is argued by constructing structures homologous to Möbius bands.

Let Ω be a simple closed curve. Consider any two distinct points A,B $\in$ Ω chosen from Ω to form a point pair (A,B), and we define a mapping:

$$f(A,B) = (x,y,z) \tag{1}$$

where (x,y) are the coordinates of the midpoint of the segment AB and z is the length of the segment [4]. This mapping converts the geometric information of the point pair on the 2D curve into a point in 3D space [5], and the visualisation of this mapping process can help understand the topological connection between point pairs .

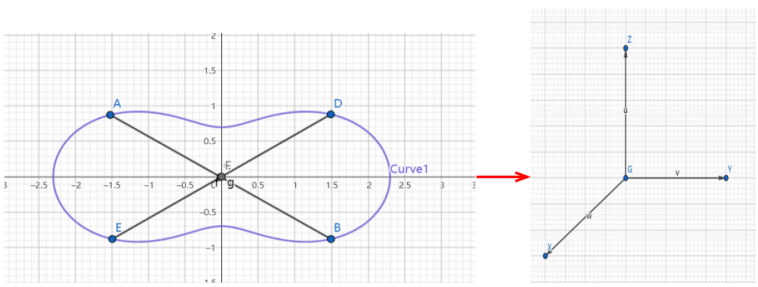


Figure 2. Point pairs (A,B) on simple closed curve

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In other words, the mapping f maps a point pair (A,B) on a curve to a point in 3D space, reflecting the geometric structure of "connected point pairs"

Let a:[0,1] $\rightarrow\mathbb{R}^2$ a:[0,1] $\rightarrow\mathbb{R}^2$ be a simple closed curve where a(0)=a(1) and there is no self-intersection on (0,1).

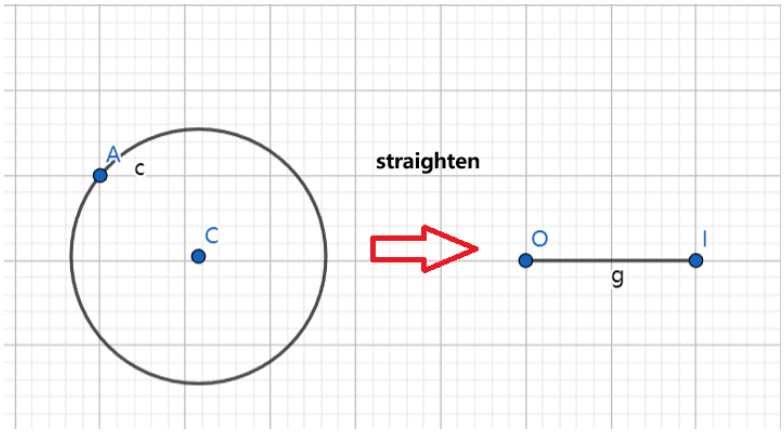


Figure 3. Parametric simple closed curve

We consider the point-to-space:

$$P = \{(a,b) \in A \times A \mid a \neq b\} \quad P = \{(a,b) \in A \times A \mid aA = b\} \tag{2}$$

Consider the mapping:

$$f : P \to \mathbb{R}^3, \quad f(a,b) = \text{line } ab \tag{3}$$

i.e., mapping each pair of points to their connected line segments in 3D space.

Whether there exists some f such that f (a,b) and f (c,d) never cross each other is explored.To study the topology of this point-pair space, we consider the parameter space [0,1] × [0,1], where each pair (s,t) denotes the point pair (a(s),a(t)). Removing the diagonal {(s,s)} means avoiding self-connections at the same point; if we consider (a,b) and (b,a) as unordered pairs of points, then in the parameter space we need (s,t)~(t,s); this is equivalent to "flipping and gluing" in a square.

We form the topology of a Möbius strip by gluing the region of [0,1]×[0,1] with the diagonal removed along the boundary in a flip-flop fashion.

This shows that the point-to-space is a Möbius band in the topological sense (boundary removed diagonal) and cannot be flatly embedded in 3D space. A known topological fact is started from: the boundary of the Möbius band is an undirected curve that cannot be embedded in R^3 without self-intersecting. If one tries to map the boundary of the Möbius strip as a continuous curve and keep its topology, there must be some regions where the "preimage" falls in the same position.

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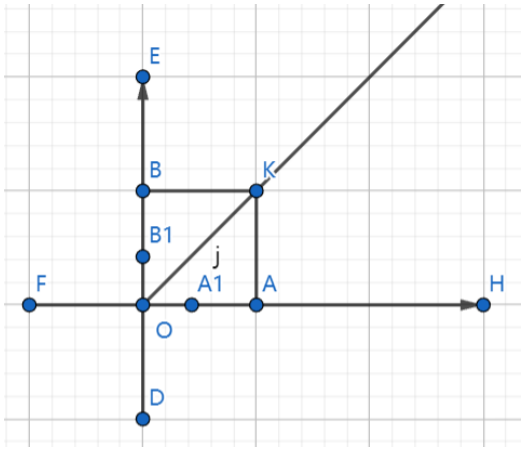


Figure 4. 3D point-pair mapped segments: self-intersection

Thus: any mapping f(a,b) from the point-to-space P to $\mathbb{R}^3$; if f is required to be continuous and satisfy the following criteria

f(a,b)=f(b,a)

Then, since the point pair space is a Möbius strip, the continuous mapping f necessarily triggers a mapping overlap; i.e., there exist different point.

Pairs (a,b) and (c,d) such that :

$$f(a,b) = f(c,d) \tag{4}$$

Self-intersection is inevitable, and the proof is complete.The overlap phenomenon of line segment mappings f(a(s),a(t)) and f(a(u),a(v)) in 3D space (i.e., the unavoidable self-intersection) is presented in Figure. 4

The process of forming a Möbius strip by gluing (s,t) and (t,s) in the parameter square, and the intersection caused by its 3D embedding, is visually explained in Fig. 5.two points (a,a ′) on a curve are connected; this corresponds to the gluing of (s,t) and (t,s) in a square; the resulting structure is a Möbius strip; when trying to embed it in 3D space as a "line image", intersections and overlaps will inevitably occur.

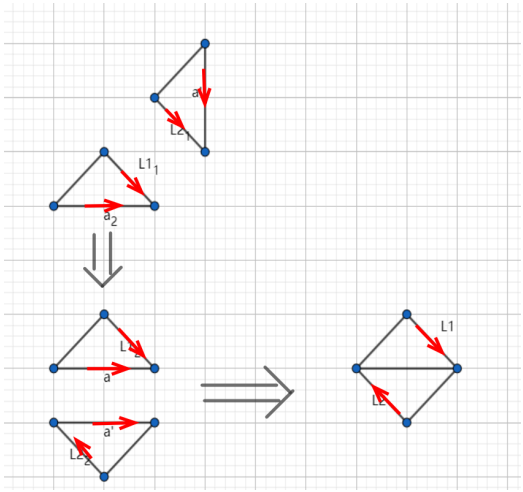


Figure 5. Intersection in mapping

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This construction is also analogous to the problem of "intersection-free embedding" of intersection maps, which is used in topological graph theory to prove that certain maps (e.g. $K3,3$) are not planar.

Definition 1 (Point-to-point mapping):Let Ω be a simple closed curve in the plane, i.e., a continuous non-self-intersecting path that meets at its end. For any two distinct points A, B ∈ Ω, define a mapping:

$$f(A,B)=(x,y,z) \tag{6}$$

where (x, y) are the coordinates of the midpoint of the line AB and z is the Euclidean length of AB.

“The distribution of 2D point pairs (A,B) on the simple closed curve Ω and their initial geometric characteristics are shown in Figure. 2.”This mapping maps the point pair (A, B) on the curve to a point in $\mathbb{R}^3$, denoted as "the image of the line segment corresponding to the point pair". Consider the set of all point pairs and their mapping images, which we call the image structure of the line segments of the curve.

Definition 2(Point pair space):Let the parametric curve a: [0, 1] → $\mathbb{R}^2$ describe the curve Ω, where a(0) = a(1). “How the parametric function a(s) describes the continuous non-self-intersecting property of Ω (with a(0)=a(1)) is demonstrated in Figure. 3.”Define its point pair space as:

$$P=\{(s,t)\in[0,1]\times[0,1]\mid s\neq t\} \tag{7}$$

denotes all non-overlapping point-pair parameters. If (s, t) and (t, s) are further considered as the same unordered point pair, an equivalence relation needs to be introduced on the parameter space:

$$(s,t)\sim(t,s) \tag{8}$$

We can interpret this equivalence space as the topological space obtained by flipping and gluing the unit square along the main diagonal after removing the diagonal.

Definition 3:(Möbius band):The Möbius band is a classical two-dimensional undirected manifold, often obtained by the following construction:

- from a unit rectangle [0,1] × [0,1].

-bonding (i.e., flip-bonding) the edges (0, y) to (1, 1-y) to form a closed band surface.

The topological properties of this structure include:

1.There is only one boundary component;

2.Is not orientable;

3.Is not embeddable in the topological sense $\mathbb{R}^3$ without self-intersection.

3. Conclusion

3.1. Convex curves: square embedding & constraints

Evidence in favour of the conjecture "there is always an interior square in any simple closed curve" is given by means of the first class of constructive methods.In the case of convex curves, the required squares are constructed by means of midpoint trajectories and continuity methods, and the rotation parameter leads to an application of the median theorem, which yields a sufficiency for the existence of squares. This method may provide a route to problems in more general forms [6].

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3.2. General curves: interior rectangles, proof & adaptation

In trying to enumerate all pairs of connections to construct an interior square, some sort of "construction path intersection" is inevitable. Such an intersection suggests that the "square structure" may serve as a critical point or a stable form for these paths, and thus provides topological logic to support the existence of squares.In this paper, based on the original method of geometric construction and median theorem, the topological space perspective is further introduced to construct a three-dimensional embedding mapping based on point pairs (A,B), which is proved to be topologically equivalent to the Möbius strip by parameter space bonding analysis. From the property that Möbius bands are not embeddable in 3D space without self-intersection, it follows that any natural mapping of point pairs on simple closed curves into 3D line segments cannot avoid mapping intersections or overlaps. This conclusion reinforces the idea that "there must be a symmetry construction between certain pairs of points" from the point of view of spatial structure, i.e., it provides a new path of proof for the existence of the endohedral square [7].

4. Future prospects

Future directions for further exploration include: 1. Higher dimensional generalisation: the point pair space is extended into symmetric maps on $S^1 \times S^1$ or T^2; 2. Algebraic topological methods: tools such as fundamental groups and homotopy groups are tried to be introduced... 3. Generalised construction: the mapping f (A,B) is extended into a closed graphical mapping with a four-point structure (A,B,C,D) to describe more complex endomorphisms (A,B,C,D) in a unified way. describe more complex embedding patterns (e.g., rectangles, rhombuses, parallelograms, etc.).

Author Contribution

Jingran Shao and Haoxuan Li: Conceptualization, Methodology, Data curation, Writing- Original draft preparation, Visualization, Investigation.

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