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Study on Rainbow Ball Puzzle: From 12 Hole to 20

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Abstract. The Rainbow Ball is a spherical sliding puzzle with one empty slot. We analyze the 12-hole and 20-hole versions using graph theory to model moves as edges and group theory to depict different initial states as permutations. For the 12-hole puzzle, adjacent transpositions and a long cycle generate the full symmetric group S11, proving every arrangement is solvable. In the 20-hole puzzle, moves form 5-cycles—only even permutations—so half of all configurations are unreachable. We suggest adding a channel to create a 3-cycle, which would restore full solvability.

Keywords: graph theory, cycle, permutation, model

1. Introduction

Sliding puzzles have fascinated mathematicians for over a century. One of the most famous examples is the Fifteen Puzzle, created by Sam Lloyd in the 1870s. It consists of a 4x4 grid with fifteen tiles numbered 1 through 15, as well as a single empty square. Each move on the puzzle consists of sliding a tile into the vacant place to rearrange the puzzle. However, not every puzzle state — a specific arrangement of the numbers and the empty hole — is reachable by legal moves: for example, swapping two tiles was proven by many to be impossible [1,2].

A rainbow ball is a colorful twist on classic sliding puzzles: usually there are eleven balls of different colors rotating around a spherical shell with a single empty slot. Much like Sam Lloyd's infamous Fifteen Puzzle, it invites the question: which arrangements of the balls are attainable? To find the answer to this question, we will examine many different forms of the rainbow ball through the lens of modern algebra.

As we can see in figure 1, a 12-hole rainbow ball puzzle is a sphere with 12 holes and 11 colored balls. The goal is to move each ball into the hole with the matching color outline, leaving the 12th hole empty. Each hole is adjacent to 5 others. A move consists of sliding a ball from a neighboring space into the empty hole, leaving its original hole empty.

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Figure 1. The 12 hole rainbow ball

Just like figure 2, the 20-hole Rainbow Ball is an intriguing puzzle that expands on the classic spherical puzzle concept. It consists of twenty holes arranged in a spherical pattern, nineteen of which are filled with balls of varying colors, leaving one hole empty.



Figure 2. The 20 hole rainbow ball

To analyze every possible puzzle state, we first view the puzzle as a graph. Model each hole as a numbered vertex unless it is empty, and connect two vertices with an edge if their holes are adjacent on the actual puzzle. In the common 12 ball version, these vertices and edges form a 5 regular graph (Figure 2). This way, any move switching a numbered vertex and the blank corresponds to a legal move on the actual puzzle. A sequence of moves then becomes a walk on this graph, and any reachable state on the graph is reachable on the puzzle [3].

From a second viewpoint, each configuration of the colored balls is a permutation of the set of numbers {1, 2, 3,..., n-2, n-1} in which n is the number of holes. This set of all configurations forms a subgroup of the symmetric group Sn-1. Our central question is: which subgroup is it?

To answer this question, we look for a generating set of simple permutations, typically two cycles (transpositions) or 3 cycles, realized by specific move sequences. Finding these generating permutations will allow us to determine the proper subgroup realized by the puzzle. If such generators cannot be constructed, then the reachable states will likely form a smaller and less neatly described subgroup.

Before we dive into the Rainbow Ball puzzle, let’s recall a few essential ideas from group theory and graph theory.

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2. Group theory's basic concepts

For a sliding puzzle, the relevant groups are permutation groups, which can reflect how the pieces are rearranged. In a finite set X = {1, 2, ..., n} , its permutation is a bijection from X to itself, and the symmetric group $S_n$is formed by the set of those [4].

We will express permutations in cycle notation: for example, (1 4 3) means 1 => 4, 4 => 3, 3 => 1, with all other elements remaining fixed.

A cycle of length two, (a b), is known as a transposition.

To multiply two permutations $\sigma,\tau \in S_n$, we simply apply one and then the other.

$$(\sigma\tau)(x) = \sigma(\tau(x))$$

When we write permutations in cycle notations, multiplications follow the same rule.

To compute (1 3) (2 4) (1 2 3), we apply (1 2 3) first to obtain {2, 3, 1, 4}, then (2 4) to obtain {2, 4, 1, 3}, and finally (1 3), resulting in {1, 4, 2, 3}.

Every permutation of n elements can be written as a product of transpositions [4]. for example:

(1 3 4 2) = (1 2) (1 4) (1 3)

Because the product of an even number of transpositions is called an even permutation, the set of all even permutations form the alternating group An $\subseteq S_n$.

Sliding puzzles will naturally define some subgroup of S_n: since each legal move rearranges the n – 1 pieces and the blank. The puzzle group is a set of all permutations obtainable by some sequence of legal moves. For instance, the classic Fifteen Puzzle has puzzle group A_{15}; only even permutations of the tiles are achieveable, proving that a single transposition of the tiles 14 and 15 is impossible.

3. Graph theory's basic concepts

Graph theory is a branch of mathematics concerned with the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph is a collection of vertices (or nodes) connected by edges (or arcs) [5].

Graph: A structure consisting of a set of vertices and a set of edges connecting these vertices, typically denoted as G=(V,E), where V is the set of vertices and E is the set of edges.

- Directed Graph (Digraph): A graph where edges have orientations.
- Undirected Graph: A graph where edges do not have orientations.
- Path: A sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence [5].
- Cycle: A path that starts and ends at the same vertex.
- Connected Graph: A graph in which there is a path between every pair of vertices.
- Tree: A special kind of connected graph that has no cycles and a unique path between any two vertices

4. Simple proof of 12-hole rainbow ball

The introduction has been covered before, The rainbow ball puzzle is a sphere with 12 holes and 11 colored balls. We can see Figure 3, In order to show the possibility of every permutation of the small

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colored ball on the 12-hole Rainbow Ball. We decide to use the symmetric group to prove this process.

Guarantees that the symmetric group S_{12} is wholly generated by 2-cycles (transpositions). Then, if our move set can realise a judiciously chosen subset of transpositions, like adjacent swaps along a single track (like in figure 4). It means that any even or odd permutation of the balls can be assembled by composition, so the 12-hole rainbow ball becomes fully solvable.

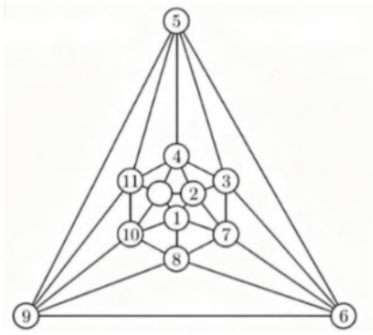


Figure 3. 12-hole rainbow 2D graph

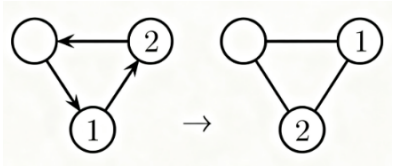


Figure 4. Adjacent swaps along a single

The proof : We need to prove that the symmetry group Sn can be generated by (1,k) form of 2-cycle.

If a=1 or b=1, then (a,b) is form of (1,k), so it does not require further decomposition.

If a,b∈[n]{n≠1},we need to make (a,b) divide into several 2-cycle of from (1,k).

This goal can be achieved through the following equation:

$$(a, b) = (1, a)(1, b)(1, a).$$

So that, any a-cycle in Sn can be represented as a product of cycles of the form (1,k). It means that Sn can be generated by 2-cycles.

Then, in the puzzle group, we can create a special “long cycle” that is showed in figure 5, specifically:

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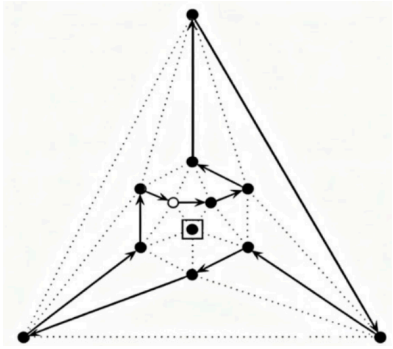


Figure 5. The process of transition

1. 11 is shifted rightward and downward into the empt slot,
2. 10 is shifted upward to the 11 slot,
3. 9 is shifted upward and rightward into the 10 slot,
4. 8 is shifted downward and leftward into the 9 slot,
5. 7 is shifted downward and leftward into the 8 slot,
6. 6 is shifted upward and leftward into the 7 slot.
7. 5 is shifted downward and to the rightward into the 6 slot,
8. 4 is shifted upward into the 5 slot
9. 3 is shifted upward and to the leftward into the 4 slot,
10. 2 is shifted upward and to the rightward into the 3 slot,
11. Finally, 11 is moved rightward into the 2 slot

In conclusion,Since adjacent transpositions generate S_11, the puzzle group contains every permutation of the eleven balls. Consequently, starting from any initial site , the balls can be moved into every desired order while the empty slot occupies any prescribed hole; so that the 12-hole Rainbow Ball is fully solvable.

5. Solvability of 20-hole rainbow ball

There is different situation happened in 20-hole rainbow ball. In this puzzle, each track move is a 5-cycle, which is showed in figure 6. Since it's not a 3-cycle, the previous theory is not applicable to this puzzle. Moreover, the 5-cycle is an even permutation, so odd target permutations lie outside this group and can never be reached, leaving at least half of all possible ball arrangements ermanently unsolvable.

From another perspective, the 2-cycly enables the two points to swap their positions with each other (123)=(13)(12).Obviously, The 5-cycle won't work. (1 2 3 4 5)=(1 5)(1 4)(1 3)(1 2).

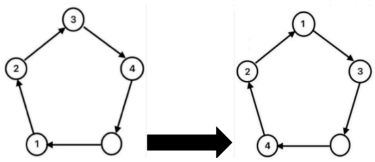


Figure 6. Diagram of the transformation

6. How to solve this problem

From the previous 12-hole rainbow ball, we can conclude that we need an a 3-cycle and then create a long cycle in order to solve this type of puzzle. In order to solve the first difficulty came up that is

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odd permutation lie outside this group, we decide to to add a channel within 20-hole rainbow ball to transform one 5-cycle within it into a 3-cycle.(As in figure 7)

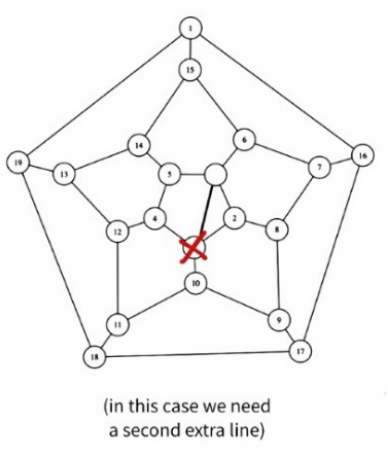


Figure 7. Building a new channel

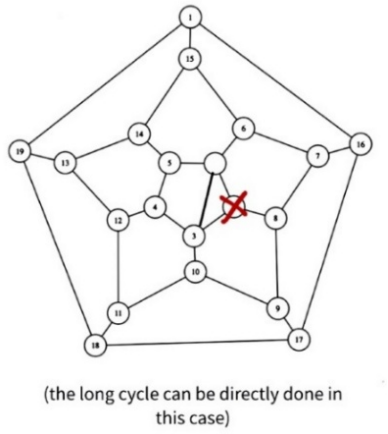


Figure 8. Opposite to the channel

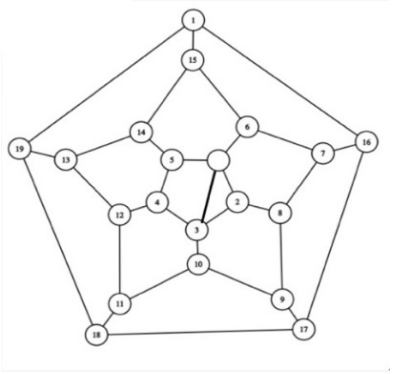


Figure 9. Dot on the channel

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However, when we think about it, we find that there are two ways of selecting the dot being desbited for the rest to build up a long cycle. So that the balls can be arranged into the correct order one by one, passing through the 3-cycle. And which dot will be selected can be divided into two situations.

Just like in figure 8, if the dot is opposite from the channel, then we can find the long cycle

1. 5 is moved right into the empty slot,
2. 4 is shifted upward and rightward to the 11 slot,
3. 12 is shifted upward and rightward to the 4 slot,
4. 13 is shifted downward and rightward into the 12 slot,
5. 14 is shifted downward and leftward into the 13 slot,
6. 15 is shifted downward and leftward into the 14 slot,
7. 6 is shifted upward and rightward into the 15 slot.
8. 7 is shifted upward and leftward into the 6 slot,
9. 8 is shifted upward and rightward into the 7 slot,
10. 9 is shifted upward into the 8 slot,
11. 17 is shifted upward and to the leftward into the 9 slot,
12. 16 is shifted downward and leftward into the 17 slot.
13. 1 is shifted downward and to the rightward into the 16 slot.
14. 19 is shifted upward and rightward into the 1 slot.
15. 18 is shifted upward and leftward into the 19 slot
16. 11 is shifted downward and leftward into the 18 slot
17. 10 is shifted downward and leftward into the 11 slot
18. 3 is shifted downward into the 10 slot
19. Finally, 5 is moved into the 3 slot

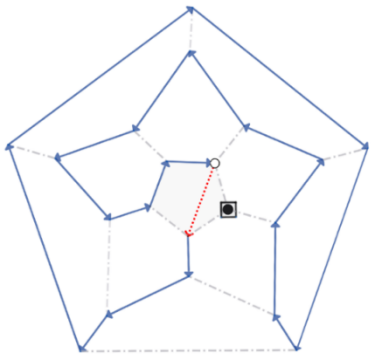


Figure 10. Choose the dot opposite from the channel

So the long cycle is: empty-5-4-12-13-14-15-6-7-8-9-17-16-1-19-18-11-10-3-empty.

This enables the rainbow ball to be solved.

Now comes to the second situation, in which the dot is on the additional channel, as in figure 8.

As we have 'deleted' the 3rd slot, four channels are despited for sure, and the 6 blue lines can be ensured. Then how to choose the rest channnels in order to build a 19-cycle needs discussion basing on spcific situation.

[Situation 1], if we choose to connect 5 and empty slot, then the two channels of slot 1, slot 15, slot 6, can be decided. However , the situation needs further classification,

if we connect 13 and 19 slot, the rest channels can be easily decided, in which 12 slot does not have enough channels around it, and the situation fails. As in figure 11.

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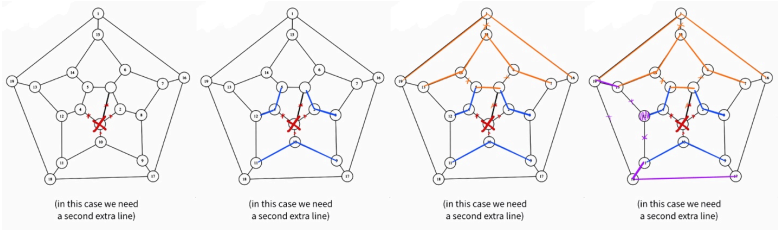


Figure 11. Passing 13 -19

if we choose not to connect 13 and 19 slot, which means 12-13 and 18-19 are needed instead. Obviously, we can find out it still fails to work. As in figure 12

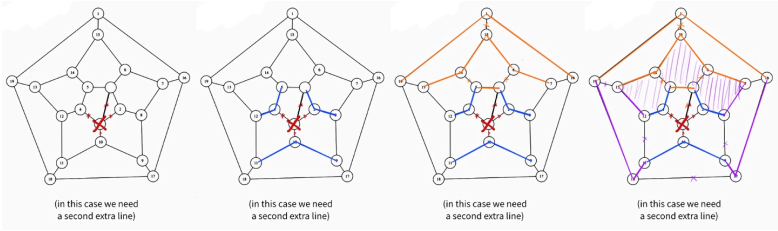


Figure 12. Not passing13 -19

[Situation 2] if we not to connect 5 and empty slot, in which one of 15-14 and 6-15 are needed.

1) if we connect 14-15, then 12-13, 13-19, 11-18 are needed for sure. But it still makes 2 possible situations after that. Just as in figure 12, we connect 18 slot and 19 slot, after the rest being confirmed, 17 slot is completely isolated, and the situation fails. As in figure 13

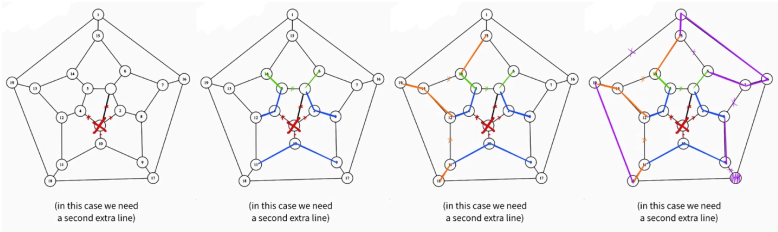


Figure 13. Passing 18-19

2) basing on the orange channels, if not using channel 18-19, as it goes there will be another isolated slot 7, the situation also fails. As in figure 14.

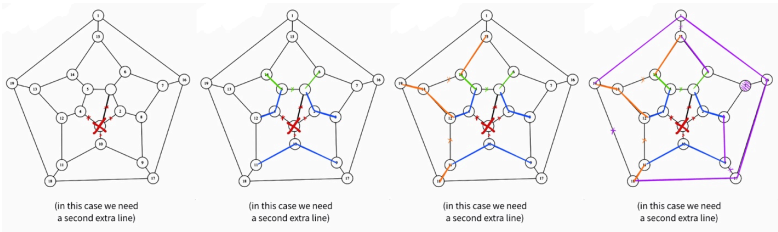


Figure 14. Not passing 18-19

3) if we connect 6 slot and 15 slot, it finally turns out that the solution is not sticking. The 11 slot have 3 connecting channels around it, which is not allowed, and the situation also fails. As in figure 15

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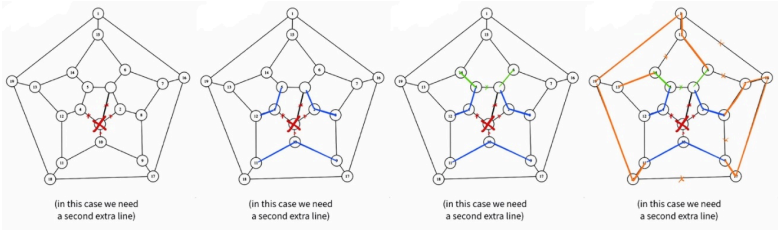


Figure 15. Passing 6-15

In summary, it's impossible to build a 19-cycle if we choose to despite one of the slots on the endpoint of the additional channel as we can see in all the situations above.

7. Conclusion

The Rainbow Ball puzzle provides a three-dimensional analogue of the classic Fifteen Puzzle, and illustrates how modern algebra can reveal the structure of sliding puzzles.

We were able to determine the puzzle groups for two versions of the toy through modeling each hole as a vertex of a graph and each move as a permutation of the colored balls.

For the 12-hole Rainbow Ball, a small set of adjacent transpositions and a single long cycle can generate the entire symmetric group S_{11}. This means that every permutation of the eleven balls is attainable, and the puzzle is completely solvable from any initial configuration, unlike the classic Fifteen Puzzle.

The 20-hole Rainbow Ball, on the other hand, behaves quite differently. Since each hole is triangular and only connected to 3 others in a pentagonal configuration, its natural move set consists of 5-cycles, which are even permutations. Because odd permutations cannot be expressed as a product of these moves, at least half of all theoretical arrangements are eliminated.

We outlined a potential modification that could restore full solvability: adding an auxiliary connection to create an effective 3 cycle.

These results show the impact modern algebraic, and graph theoretic methods had on the classification of sliding puzzles.

Modeling the puzzle as a graph clarifies which local moves are possible, while group theory translates those moves into generators of a permutation group. Together, they provide a complete framework for deciding when a seemingly simple toy is, in fact, mathematically solvable.

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