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Graphing a Rubik's Cube in Music

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Abstract. The connections between the Rubik's Cube, group theory, and Cayley graphs is explored. It provides a comprehensive introduction to these concepts and their relation. The Rubik's Cube is a popular puzzle, and the collection of all its possible states constitutes a group under face rotation operations. Focusing on the Rubik's Cube group, we delve into the mathematical structure underlying this puzzle's structure. A key object of this exploration is the Cayley graph, which visually represents the Rubik's Cube group. This graph connects the Rubik's Cube with many other topics and lead to many applications. By modeling the Rubik's Cube as a random walk on its corresponding Cayley graph, we gain insights into scrambling and solving processes. Another surprising application is its connection to musical patterns through modulation graphs. This paper aims to bridge the gap between abstract algebraic concepts and their tangible applications in the Rubik's Cube.

Keywords: Rubik's Cube, Group theory, Cayley graph, Modulation, Random walk

1. Introduction

I am a Rubik's Cube enthusiast with a passion for mathematics. I am a 2-year member of the China Rubik's Cube team who has participated and won 2 competitions organized by the Shanghai Rubik's Cube Association. I also made 3×3 semi-finals in the CubingUSA Western Championship 2023.

I have also found interest in music when I started playing the piano at 4, and the violin at 7. Soon after, I started playing in an orchestra. One day, when I was listening to classical music, I came across the Mozart Dice Game, which is a composition where pre-written musical fragments are randomly assembled to created musical pieces. This inspired me to create my own version of the dice game using a Rubik's Cube, namely "Cubin' Composers", where each turn would play one measure of the musical track. In the near future, I hope to physically create the Rubik's Cube as an instrument to play music. This paper will provide the theoretical foundation for the future creation of this Rubik's Cube instrument.

The Rubik's Cube is a widely celebrated puzzle across the world. It can be a fun challenge for some people and a competitive sport for others. The puzzle involves a player restoring a scrambled cube, or bringing the colors back to their original position, with one distinct color on each face. This scrambling process is especially important to providing adequate challenge as well as ensuring fairness among all competitors. What many may not know is that using the mathematical theories of group and graph, one is able to study the structure of the cube and its process of scrambling scientifically.

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Such a study has at least three applications. Firstly, to ensure that all scrambles are challenging enough for the contestant. Secondly, to ensure that different scrambles have a similar level of difficulty. One way of measuring the difficulty of a scramble is the length of its optimal solution. The God's number is defined to be the minimum number of turns required to solve the puzzle from any starting position [1]. In 1995, Michael Reid established an upper bound of 29 for God's number and identified a configuration that could be solved in 20 moves, thereby raising the lower bound to 20 and reducing the gap to nine moves. Later, in 2008, Tomas Rokicki together with John Welborn further tightened the upper bound to 22. This effort culminated in 2010, when Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge rigorously demonstrated that God's number is exactly 20. It was also found that around 66.4% of all scrambles have an optimal solution of 18 moves. Last but not least, graphs show up in many other art forms such as modulation graphs in music. It is plausible that the highly symmetric structure of a Rubik's Cube can give rise to new musical patterns.

In this article, we outline a mathematical framework of the Rubik's Cube using Cayley graphs. Scrambling is modeled as a random walk on the Cayley graph associated to the Rubik's Cube. Thus, the study of Rubik's Cube scrambling becomes a problem in random process, which has been widely studied [2-4]. This builds a bridge between the famous puzzle and the existing mathematical literature. We believe the connection we draw is a valuable addition to both worlds.

In section 2, we will learn how to describe rotations, as well as naming specific pieces on the cube. Section 3 provides a quick introduction to group theory and some applications on the Rubik's Cube. In section 4, we introduce Cayley Graphs and how they model the configurations of the Rubik's Cube. Then, in section 5, we will talk about how the act of scrambling a Rubik's Cube is like a random walk on its Cayley graph. Last but not least, section 6 explores tonal modulation in musical composition based on the Rubik's Cube.

2. Rubik's Cube

2.1. A brief history

In 1974, Hungarian architect Erno Rubik built the first version of his cube and used it as a tool to teach 3D geometry. Rubik patented his cube mechanism, and in 1977, the puzzle was sold over Hungary, initially known as the Magic Cube. Later, in 1980, Rubik partnered with the Ideal Toy Company and brought the puzzle to the global market. It was also renamed to "Rubik's Cube" [5]. In 1981, David Singmaster published the first analysis for the Rubik's Cube, Notes on Rubik's 'Magic Cube' [6], where he introduced 'Singmaster Notation' to describe different rotations on the cube, a notation system still used today. By January 2024, global sales of Rubik's Cubes had reached roughly 350 million units, underscoring their status as one of the most recognizable and commercially successful puzzles worldwide.

2.2. Structure and mechanics

A brief overview of the standard Rubik's Cube is given. Readers may refer to [7] for a detailed account. The standard 3x3 Rubik's Cube consists of 27 smaller cubes, commonly known as "cubies". A systematic way to refer to individual cubies is by naming them based on the locations of the cubies. The pieces located at the center of each face are referred to as center cubies. There are six such pieces, each exposing a single face. Pieces with two visible faces are known as edge cubies,

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and the cube contains twelve of them. Pieces with three visible faces are called corner cubies, of which there are eight.

To name the 6 faces of the Rubik's Cube, we follow David Singmaster's convention [7] and assign a letter to each of the faces, including right (R), left (L), up(U), down(D), front(F), back(B).

Cubies are labeled by their visible faces; for example, the upper-left-front corner cubie is written as LUF (or UFL or FLU). The edge located in the front right is notated as FR (or RF).

Another key concept is that of "cubicles." Like cubies, cubicles are named according to face positions, but they refer to fixed locations on the cube rather than the pieces themselves. For example, in the solved state, each cubie occupies the cubicle with the same name. When the cube is rotated, the cubies change positions while the cubicles remain fixed. In other words, cubicles are stationary, whereas cubies move.

2.3. Combinations

It is important to note that when any face of the cube is rotated, all center cubies remain in their original cubicles. Furthermore, a corner cubie always stays in a corner cubicle and an edge cubie is always kept in an edge cubicle. This means that a corner cubie may never be in an edge cubicle, nor can an edge cubie ever be in a corner cubicle.

According to this information, the number of valid configurations for a 3×3 Rubik's Cube can be calculated. Since there are 8 corner cubies, there are 8 spots for the first corner cubie. Subsequently, there are 7 spots for the second corner cubie, 6 spots for the third, and so on. That gives $8*7*6*5*4*3*2*1 = 8!$ possible results. Each corner cubie could also be oriented in three ways. Each corner also has 3 orientations, which gives 3^8 ways to orient the corners. Similarly, the 12 edge cubies can occupy 12! possible positions. Each edge cubie has two possible orientations, resulting in 2^{12} possible edge orientations. Together, this gives $3^8 *8!$ corner configurations and $2^{12} *12!$ edge configurations, giving a total of $2^{12} *12! *3^8 *8! = 5.1 *10^{20}$ positions.

Even though all of these configurations are theoretically possible, not all of them can be achieved using regular rotations from the solved position. These are called invalid configurations. Hence, all valid configurations can be returned to the solved position using a set of moves.

One type of impossible situation is cases where a single edge cubie is flipped. Since each cubie has 2 orientations, this doubles the amount of configurations. We need to divide the total configurations by 2 to account for the overcounting. Similarly, we divide by 3 to account for cases where a single corner cubie is twisted. Finally, we divide by 2 because each edge must be an even permutation exactly where the corners are. In total, this divides the result by $2 *3 *2 = 12$, giving $4.3 *10^{19}$ valid configurations.

3. Group theory

3.1. Groups

Definition Let G be a finite or infinite set and $\circ$ be a binary operator. The pair $(G, \circ)$ is a group if

1. G is closed under $\circ$. If $a, b \in G$, then $a \circ b \in G$.
2. $\circ$ is associative.
3. There exists an identity element $e \in G$ such that $a = a \circ e = e \circ a$ for all $a \in G$
4. For every $a \in G$ there is an inverse $a^{-1} \in G$ such that $a^{-1} \circ a = a \circ a^{-1} = e$

$\circ$ denotes multiplication, e is the identity element of G, and a-1 is the inverse of a.

Example Some examples of groups are:

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1. 1. $(\{0\},+),(\mathbb{Z},+),(\mathbb{Q},+),(\mathbb{R},+),(\mathbb{C},+)$
2. 2. $SO\left(2\right)=\left\{\begin{pmatrix}x & y\\ 0 & 1/x\end{pmatrix}:x,y\in\mathbb{R},x^2+y^2=1\right\}$

(SO stands for "special orthogonal")

3. 3. The symmetric group Sn is the set of permutations of n ordered objects denoted by $=(0,1,2,3,\ldots,n)$. Multiplication is defined to be composition of permutations. Identity is the trivial permutation where nothing is done.

3.2. Making the Rubik's Cube into a group

A "move" can be defined as a sequence of turns. The set of all possible moves in a Rubik's Cube can be structured as a group, denoted $(G,*)$, where each element of G corresponds to a distinct move. For instance, a 90° clockwise rotation of the right face is represented as R . Two moves are considered identical if they produce the same resulting configuration; for example, rotating a face 180° clockwise is equivalent to a 180° counterclockwise rotation. The group operation is defined such that if S_1 and S_2 are two sequences of moves, then S_1*S_2 is the sequence of moves where you first do S1 and then do S_2 (e.g, $R*U$ denotes RU).

To prove that this is a group, we need to show the 4 properties in section 3.1

1. $\mathbb{G}$ is closed since if S_1 and S_2 are two sequences of moves, then S_1*S_2 is a sequence of moves as well.

2. Let e be the "empty" move, a move that does not change the configuration at all. Then $S*e$ indicates to first do S, then do nothing. This is the same as just doing S, so $S*e=S$

3. Let C be a cubie. Denote $S(C)$ for the cubicle that C ends up after the sequence S is performed. Let's see what two sequences of moves do to a cubie. If S_1 and S_2 are two moves, then S_1*S_2 is a sequence where we first do S_1 and then do S_2 . The sequence S_1 takes C to $S_1(C)$, and the sequence S_2 takes it to $S_2(S_1(C))$. Hence, $(S_1*S_2)(C)=S_2(S_1(C))$.

Let $S(C)=C_1$. The sequence S can also be reversed and notated as S' . By definition, $S'(C_1)=C$. Then,

$$S*S'=S'(S(C))=S'(C_1)=C \tag{1}$$

Since the sequence $S*S$' has done nothing to C, $S*S'=e$ Therefore, S′ is the inverse element of S.

4. To show associativity, we simply have to show that $(S_1*S_2)*S_3$ and $(S_1*S_2)*S_3$ yield the same cube configuration. We can further use the property $(S_1*S_2)(C)=S_2(S_1(C))$ to show this:

$$(S_1*S_2)*S_3=((S_1*S_2)*S_3)(C)=S_3((S_1*S_2)(C))=S_3(S_2(S_1(C))) \tag{2}$$

$$S_1*(S_2*S_3)=(S_1*(S_2*S_3))(C)=(S_2*S_3)(S_1(C))=S_3(S_2(S_1(C))) \tag{3}$$

Therefore, $(\mathbb{G},*)$ is a group.

3.3. Homomorphisms

Definition Given two groups $(G,\circ)$ and $(H,*)$, a map $\phi:G\to H$ is referred to as a homomorphism if it preserves the group operation from G to H , if for all $a,b\in G$

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$$\varphi(a \circ b) = \varphi(a)*\varphi(b) \tag{4}$$

Example Some examples of homomorphisms are:

1. Consider two $n \times n$ matrices A and B, the mapping $\det : F^{n*n} \to F$, from $(A, \circ)$.

3.4. Kernel and image

Definition Consider $\phi : G \to H$ as a homomorphism from $(G, \circ)$ to $(H, *)$, and let e_G, e denote the respective identity elements. The set

$$im\ \varphi = \{\varphi(a) : a \in G\} \subseteq H \tag{5}$$

is the image of φ, and

$$ker\ \varphi = \{a \in G : \varphi(a) = e\} \subseteq G \tag{6}$$

is the kernel of φ.

To prove that $(im\ \varphi, *)$ and $(ker\ \varphi, \circ)$ are groups, we need to show the 4 properties in section 3.1.

The kernel satisfies all four axioms, as shown in the following:

1. closed

$$\begin{matrix} \varphi(a) & = e_H, \varphi(b) = e_H \\ \varphi(ab) = \varphi(a)\varphi(b) & = e_H \end{matrix} \tag{7}$$

2. associative

$$\begin{matrix} (\varphi(a) *\varphi(b)) *\varphi(c) = e_H^2 *e_H = e_H \\ \varphi(a) *(\varphi(b) *\varphi(c)) = e_H *e_H^2 = e_H \end{matrix} \tag{8}$$

3. identity

By definition,

$$\varphi\left(e_G\right) = e_H \tag{9}$$

4. inverse

$$\varphi(a) *\varphi^{-1}(a) = e_H *e_H = e_H \tag{10}$$

The image also satisfies all four axioms, as shown in the following:

1. associative

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$$\begin{aligned}(\varphi(a)*\varphi(b))*\varphi(c) &= \varphi(ab)*\varphi(c)\\ &= \varphi((ab)c)\\ &= \varphi(a(bc))\\ &= \varphi(a)*\varphi(bc)\\ &= \varphi(a)*(\varphi(b)*\varphi(c))\end{aligned} \tag{11}$$

2. identity
By definition,

$$\varphi(e_G) = e_H \tag{12}$$

3. inverse

$$\varphi(a)*\varphi^{-1}(a) = \varphi(aa^{-1}) = \varphi(e_G) = e_H \tag{13}$$

3.5. Subgroups

Definition A subset H of a group G that is closed under the group operation of G is called a subgroup, denoted $H \leq G$. If $H \neq G$, it is referred to as a proper subgroup.

Example Some examples of subgroups are:

1. $(\mathbb{Z},+) < (\mathbb{Q},+) < (\mathbb{R},+) < (\mathbb{C},+)$
2. Fpr $\mathrm{n} \in \mathbb{N}, (n\mathbb{Z},+) \leq (\mathbb{Z},+)$

Note: $(\mathbb{R}^*, *)$ is not a subgroup of $(\mathbb{R},+)$ because they do not share the same group operation $(\mathbb{R}^* = \mathbb{R}\backslash\{0\})$.

There are also many subgroups of the Rubik's Cube group $(\mathbb{G}, *)$. For instance, let $(\mathbb{H}, *)$ be the group obtained with only 180◦ turns (i.e. $\left[R^2, L^2, U^2, D^2, F^2, B^2\right]$) [8]. All elements of $\mathbb{H}$ are obtainable through $\mathbb{G}$, while not all elements of G are contained in H. Intuitively, multiples of 90° turns can generate 180° turns, while moves solely composed of 180° turns cannot obtain all combinations of 90° turns. So, $\mathbb{H} < \mathrm{G}$.

4. Cayley graph

4.1. Definition and construction

A graph Γ is consists of vertices and edges. Mathematically, these are specified as follows:

- A finite set V, referred to as the vertices;
- A finite set E, referred to as the edges;
- A function ζ that associates each edge e with one or two vertices v. The set $\zeta(e)$ is called the endpoints of e.

The graph Γ can be constructed from the above data. Begin with a set of points representing the vertices and a set of segments representing the edges. For each edge e, the initial vertex of its segment is identified with one vertex in $\Gamma(e)$, and the terminal vertex is identified with the other vertex in $\Gamma(e)$.

In a directed (or oriented) graph, each edge has a direction, running from the initial vertex $\iota(e)$ to the terminal vertex $\tau(e)$. In other words, there are two mappings

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$$\begin{aligned} \iota &: E \to V \\ \tau &: E \to V. \end{aligned} \tag{14}$$

Figure 1 gives an example of a graph. It consists of three vertices, $V = \{v_1, v_2, v_3\}$, and three edges, $E = \{e_1, e_2, e_3\}$ with

$$\begin{aligned} e_1 &\to \{v_1, v_2\} \\ e_2 &\to \{v_2\} \\ e_3 &\to \{v_2, v_3\} \end{aligned} \tag{15}$$

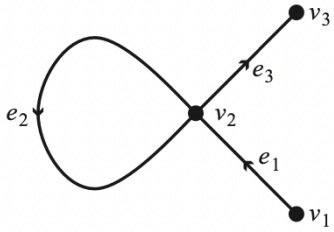

Figure 1. Example of a graph

4.2. Generators

Let G be a group and S a subset of G. A generating set S is a subset such that every element of G can be expressed as a finite combination of elements from S and their inverses.

The subgroup generated by S, denoted $\langle \mathrm{S} \rangle$, is the smallest subgroup of G containing all elements of S. If $\mathrm{H} = \langle \mathrm{S} \rangle$, we say that S generates H, and the elements of S are called generators of H. When S contains only a single element x, $\langle \mathrm{S} \rangle$ is written as $\langle \mathrm{x} \rangle$ and is referred to as the cyclic subgroup generated by x; in this case, x generates the entire subgroup H.

A group may need infinitely many generators. For example, the rational numbers $(\mathbb{Q}, +)$ is not finitely generated. If S is finite, then we call $\langle S \rangle$ finitely generated.

Example Some examples of generators include:

1. $(\langle 2 \rangle, +) = (2\mathbb{Z}, +)$
2. $(\langle 2,3 \rangle, +) = (\mathbb{Z}, +)$
3. $\left(\langle (1,0), (0,1) \rangle, +\right) = \left(\mathbb{Z}^2, +\right)$
4. $\left(\langle 3 \rangle, *(\mathrm{mod}\ 7)\right) = \left(\mathbb{Z}/7\mathbb{Z}^*, *(\mathrm{mod}\ 7)\right)$
5. $(\langle R, L, U, D, F, B \rangle, *) = (\mathbb{G}, *)$ (i.e. all possible configurations of the 3*3 Rubik's Cube)
6. Proper subgroups of $(\mathbb{G}, *) : \left(\langle R^2, U^2 \rangle, *\right) < \left(\langle R^2, U \rangle, *\right) < \left(\langle R, U \rangle, *\right) < \left(\langle R, U, F \rangle, *\right) < \left(\mathbb{G}, *\right)$

4.3. Cayley graph

Let G be a group and S a generating set for G. A graph provides a geometric representation of the group. The Cayley graph is a directed graph in which

$$V = G \tag{16}$$

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$$E = G \times S := \{(g, s) : g \in G, s \in S\}. \tag{17}$$

Each edge is thus labeled by a pair (gs), where $g \in G$ and $s \in S$. The functions ι and τ are defined as follows:

$$\begin{aligned} GS &\to G \\ (g, s) &\mapsto g \\ GS &\to G \\ (g, s) &\mapsto gs. \end{aligned} \tag{18}$$

The edge labeled by pair (g, s) runs from g to gs. It is clear from the definition that the choice of generators affects the Cayley graph. The diameter of a Cayley graph is the smallest number of elements from a generator set that can be used to represent every element of the finite group.

Let a group element g be expressed as a product $s_1^{\epsilon_1} s_2^{\epsilon_2} \ldots s_n^{\epsilon_n}$, where each $s_i \in S$ and $\epsilon_i \in \{-1,1\}$. Since S is a generating set, such a representation is always possible. This defines a path in the Cayley graph, starting at the identity vertex and traversing edge s_1 forward if $\epsilon_1 = 1$ or backward if $\epsilon_1 = -1$, then proceeding along s_2, and so on, until reaching vertex g. Therefore, any two vertices in a Cayley graph can be connected by a path.

5. Scrambling as random walk

5.1. Introduction

Consider a Cayley graph of the Rubik's Cube viewed as a group. Of course, it is too enormous to picture. However, it exists and is finite. Moreover, each vertex has six arrows pointing to configurations corresponding to one elementary move.

A random walk on a Cayley graph can effectively model the process of scrambling, particularly in the context of permutations or mixing systems like shuffling a deck of cards. In this context, the vertices of the Cayley graph represent different states or configurations, and the edges represent possible moves or operations that transition from one state to another. A random walk involves selecting edges randomly, simulating the process of applying random operations. Over time, as more random moves are made, the system becomes increasingly "scrambled," eventually reaching a state where all possible configurations are equally likely.

When performing a random walk on this graph, starting from a solved configuration, the Rubik's Cube becomes progressively more scrambled with each random move. After a sufficient number of moves, the cube reaches a state where any of the possible configurations is equally likely, meaning it is thoroughly scrambled. This model helps in analyzing the mixing time, which is the number of random moves required to reach near-uniform randomness, an important concept in both solving and scrambling the cube. This number is effectively equal to the diameter of this Cayley Graph, also equal to God's number, 20. Thus, it takes 20 moves to thoroughly scramble the Rubik's Cube.

This approach is also used to analyze the efficiency of algorithms that solve the Rubik's Cube. By understanding the structure of the Cayley graph and how the random walk behaves, mathematicians can estimate the number of moves needed to solve the cube from any given state, or conversely, to scramble it effectively. For example, if the shortest path from an arbitrary configuration to the solved state is 16, then that scramble takes a minimum of 16 moves to solve. It is also possible that the scramble can be solved in exactly 22 moves. When translated to the Cayley graph, assuming

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corresponding moves, the cube follows a longer path of length 22 to reach the solved state. This application is vital in areas such as algorithm design, where the goal is to create the most efficient solvers, and in recreational mathematics, where understanding the complexity of puzzles like the Rubik's Cube is of great interest.

5.2. The search for optimal scrambling

In order to guarantee fairness, random scrambling consisting of n independent uniformly random moves is widely adopted. In this section, we design and experiment in search of an optimal number of moves n inspired by our random walk point of view.n

Firstly, let's specify what optimal scrambling means. Scrambling is modeled as random walk on Cayley graph. The restored configuration corresponds to one vertex (origin) of the graph. Scrambling is a random walk departing from this vertex. Optimal scrambling aims to get far away from origin as efficiently as possible. When n is small, there is high probability that n random moves land at a vertex distance n from origin. However, as n grows the distance becomes strictly less than n due to loop formation. A loop in a random walk describes the situation where certain sequence of walks returns to its starting point. Therefore, scrambling moves involved in a loop become futile as they do not increase the distance from origin. We define an optimal scrambling number to be the maximal n such that

$$\Pr(\text{Loop formation within n moves}) < \epsilon \tag{19}$$

where ϵ is a small positive number. In plain terms, the optimal n is the largest number of scrambling moves with low chance of loop formation. In theory, for a specific ϵ, Pr(Loop formation within n moves) can be determined analytically. However, the astronomical size of the Cayley graph prevents it in reality. As such, experimental search needs to be adopted. We design an experiment which determines optimal n for various ϵ.

Fix a number n. Take a random number generator. Before each move, reset this generator to decide which move to take. After each move, check if the configuration it lands has appeared before. If so, terminate this round since a loop has occurred.

If no loop as occurred, terminate the round after n random moves. Repeat this for 1000 rounds and tally the number of rounds m where a loop occurs. This number m/1000 provides an approximation for Pr(Loop formation within n moves).

After computing approximate Pr(Loop formation within n moves) for a number of different n, choose a cutoff probability ϵ. The largest n with Pr(Loop formation within n moves) $< \epsilon$ is the optimal scrambling number.

6. Modulation graph and the Rubik's Cube

Tonal modulation is a musical technique where the key of a piece or section of music shifts. This change in key can be done smoothly, often by using pivot chords that belong to both the original key and the new key, or more abruptly, without much preparation. Modulation adds variety and emotional contrast to music. It can be used to heighten tension, introduce new themes, or signal a transition between sections. Modulations are common: Examples of modulation can be found in "Here, There and Everywhere" by The Beatles" [9] as well as in Beethoven's Symphony No.3 ("Eroica"), 1st Movement, creating a dramatic and unexpected shift in mood. A modulation graph [9] is a visual tool used in music theory to illustrate the key changes or modulations within a piece

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of music. It maps out the tonal center shifts from one key to another. Like the Rubik's Cube, the mathematical theory behind modulation is also graph theory. This suggests a deeper connection between the Rubik's Cube and music. In particular, this section will explore novel modulation patterns born out of scrambling the Rubik's Cube.

Specifically, we label each face of a cubie by a musical note. Each turn moves some notes onto other faces. This creates a shift of certain keys, providing a possible modulation pattern. More random scrambling results in more complicated modulation pattern. Mathematically, this operation creates a connection between its Cayley graph and novel modulation graphs. It is worth noting that, because of the vast possibilities of Rubik's Cube, we are able to generate endless new modulation patterns. Its potential value in musical innovation should not be underestimated.

For the rest of the section, we present some new modulation patterns we obtained. Consider the following naming scheme:

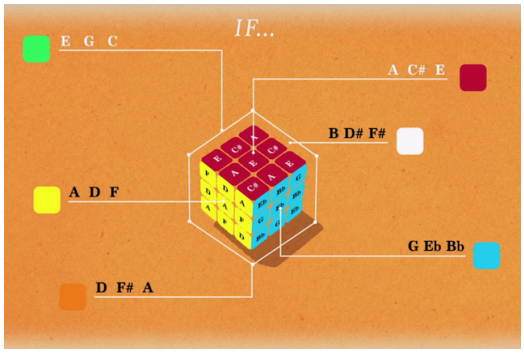

Figure 2. Cover all faces of the solved Rubik's Cube with musical notes so that each face forms a harmonious chord

A similar scheme is shown on the following cube:



Figure 3. Over entire Rubik's Cube with notes on a physical Rubik's Cube

The scramble will be FR′ F′ RURU′ R′. The algorithm that solves this scramble is RUR′ U′ R′ FRF′.

The three columns will represent the three parts on the music staff, respectively. The nine "note names" on the cubies on the upper (white) face will represent the notes. The chords may be in any inversion. For example, this is the chord for the starting configuration:

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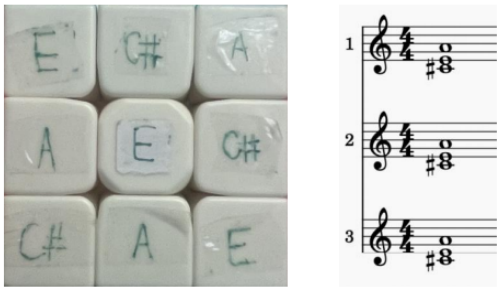

Figure 4. The chord on the upper face of the Rubik's Cube and its corresponding music staff

The score for the algorithm RUR′ U′ R′ FRF′ :

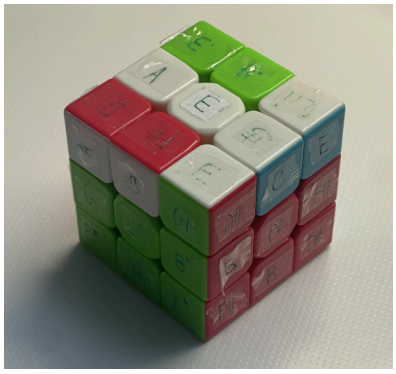

Figure 5. The Rubik's Cube with Scramble FR′ F′ RURU′ R′ applied

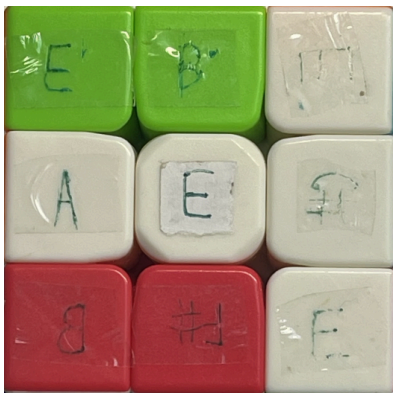

Figure 6. The chord changes of the upper face of the Rubik's Cube selected in the previous text

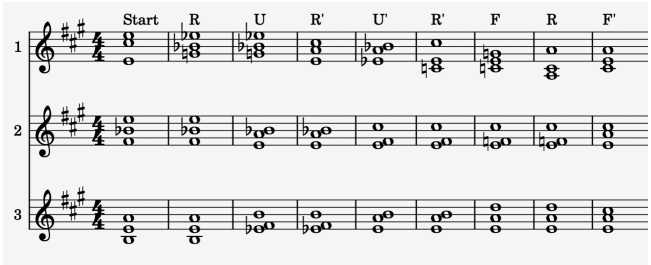

Figure 7. Chord changes during the application of scramble

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Initially, the chords are dissonant, with lots of sharps and flats, because the pieces are scrambled. Having random cubies on the upper face means that random notes are used to form the chords. The chords most notably sound more harmonious again when the cube is 1 move away from solved, because 12 cubies have already returned to their original positions. Finally, when the cube is solved, the chords sound perfectly harmonious again. So, the more scrambled the cube, the more contrasting the sound it creates.

We know that a solved cube produces a perfectly harmonious chord and a cube that is 1 move away from being solved remains close in tonality. By repeating a similar process for configurations that are two, three, or more moves away from the solved position, we can explore the tonality of the chords they produce. For example, start from a solved cube and scramble the cube using a random sequence of 7 moves. Listen and record to the chord produced by this configuration. Solve the cube, and redo this process again using different random scrambles. You may find that most 7-move scrambles sound very dissonant. You can even repeat the experiment using 9-move or 12-move scrambles, and examine the tonality of those chords.

7. Discussion

The interplay between the Rubik's Cube, group theory, and Cayley graphs presents a rich framework for understanding the mathematical structure of this puzzle. It also opens up applications of these concepts in various ways. The use of Cayley graphs to model the Rubik's Cube group provides a powerful visual and analytical tool. This helps understand and model the scrambling processes. For instance, by modeling a random walk on the Cayley graph, we can investigate the efficiency of different scrambling methods and the mixing time required to achieve a thoroughly randomized state.

Moreover, the insights gained from this graph extend beyond the Rubik's Cube. The principles of group theory and Cayley graphs have broader implications in areas such as music tonality (which we explored) [10], coding theory, and the study of complex systems. The Rubik's Cube serves as an entry point into these areas. Additionally, future work may explore algorithms for solving the Rubik's Cube, informed by the structure of its Cayley graph.

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