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Calculation of Definite Integrals with Exponential Functions Using Residue Theorem

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Abstract. The calculation of definite integrals is by no means an easy task. This paper focuses on applying the Residue Theorem and integrating two functions that cannot be solved using normal calculus. Although the general functions are real variables, there exists some limitations in real calculus that limits the possibility to solve some specific functions. The two case studies are the Dirichlet Integral and a rational function that includes trigonometry. The methodology is to first convert them into the complex plane, and then effectively using the Euler's Equation to make effective use of the properties of e-exponents e^{iz} to solve for real functions using complex analysis. More detailed process includes theories and concepts in the complex world such as poles, singularities, Cauchy's Residue Theorem, Half Lemma Theorem, etc. The results confirm that residue theorem is useful in certain cases when the definite integrals of the real function cannot be solved using traditional calculus such as integration by parts and u-substitution.

Keywords: Complex Analysis, Residue Theorem, Jordan's Lemma, Dirichlet's Integral

1. Introduction

In the study of mathematics, real number analysis is usually the start for many students. When they get to calculus, the evaluation of definite integrals is a crucial task. In normal calculus, the calculation of integral is based on the Riemann Sums and Fundamental Theorem of Calculus, relies on the bounds and antiderivatives of the function. Understanding the Fundamental Theorem of Calculus helps people to thoroughly solve some basic integrals that are practical in fields such as physics, engineering, etc. However, the range of problems that basic calculus can solve is confined. Many important integrals in physics do not have primary antiderivatives, therefore making it hard to solve using real calculus, for example the Dirichlet Integral (which will be one focus of the paper). This is when complex variables become useful.

The imaginary number i first appeared while solving cubic equations in the form $\sqrt{-1}$. Later, more explorations including the Euler's Formula, graphing of complex numbers, and complex calculus are discovered, including Cauchy Riemann Equation and Cauchy Residue Theorem. All of them are the keys to solving the case studies. These methods are very helpful in many fields nowadays, people use them in engineering, physics, etc. [1].

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Expanding from the real realm to the complex realm starts by introducing complex number i, with basic for $z = x + iy$. All subsequent formulas and theorems stem from that. This transition also converts the geometric plane from a one-dimensional number line to a two-dimensional argand plane. Therefore, a complex function is graphed in a 4-dimensional plane. The concept of analyticity is the key in complex calculus. In calculus, a function is differentiable when the limit from both sides is equal. It is similar in complex analysis, but because complex numbers are in a plane, it is defined analytic if the limit approaching from every direction is equal. Points where the function is not analytic are called singularities, and these are the keys to applying the Residue Theorem.

2. General knowledges of complex analysis

Before diving into specific case studies and applying the residue theorem, there are some basic rules for integrating complex numbers.

2.1. The Cauchy Riemann Equation

When studying derivatives of complex functions, people need to prove that the function is analytic everywhere. Unlike real variables, $f(z)$ has two dimensions x and iy. For the derivative of a complex function to exist at a point z_0, the limit as z approaches z_0 must be the same from every single direction. Unlike real calculus where the directions are only from left and right, the directions that complex numbers approach a point can be horizontal, vertical, and in a spiral. this paper defines this as analytic if the limit approaching from every direction is equal [2].

The Cauchy Riemann's Equation (C-R Equation) can be used to verify them. To apply the Cauchy Riemann's equation, first define $f(z) = u(x, y) + iv(x, y)$. For the function to be analytic, it must satisfy the C-R Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

In the first case study, $f(z) = \frac{e^{iz}}{z^2+a^2}$ is analytic everywhere except when the denominator is zero. These points are the singularities that need to be enclosed in order to guarantee analyticity before applying the Residue Theorem [3].

2.2. Classification of singularities

Singularities are places where the complex function $f(z)$ is undefined. It can be more clearly understood when it is connected to normal calculus. It corresponds to discontinuities in calculus. When performing contour integration, it is important to find the singularities and know the different types.

The first type is the Removable Singularities. Removable singularities occurs when a function appears indeterminate, usually in the form $0/0$. This occurs specifically in the second case $f(z) = sin(z)/z$ when $z = 0$. The limit still exists and can be solved after applying L'Hôpital's rule. These singularities do not get included into the residue [4].

The second type is the Poles: A pole is a singularity where the limit approaches infinity. The power of the denominators is expressed in orders. Simple poles are poles that has order I denominator. This is directly applied into the Residue Theorem.

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The third type is Essential Singularities: These singularities occur at points where it is extremely volatile. For example, $f(z)=e^{\frac{1}{z}}$ where z=0. In such singularities, the function obtains every complex value infinitely often.

2.3. Laurent Series

In standard calculus, Taylor Series is used to approximate functions using the sum of powers. However, the limitation of Taylor Series is it cannot deal with singularities. This is when the Laurent Series becomes useful, as it extends the exponents into the negative realm.

The Laurent Series at a point z_0 is:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} \tag{2}$$

The first summation is the non-negative powers. After integrating this part around a closed loop, the result is zero.

The second summation includes all the negative powers. This is the part that tells us what happens as the function approaches its poles z_0. When applying the Residue Theorem, this is the part that matters. The order of the pole is the exponent with the highest magnitude. Simple Poles defined in Classification of Singularities are poles with order 1 [5]. The Laurent Series is what connects the contour and algebraic properties of the singularity. The principal part of the function is a justification of the Residue Theorem.

3. Applications and examples

3.1. Example I

The first case is solving

$$I=\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx \tag{3}$$

where a is a constant greater than 0.

To use the residue theorem, it is important to realize the limitations of real number analysis. An integral like this will not work by applying integration by parts or u-substitution. If integration by parts is used where $u=\cos(x)$, $dv=\left(x^2+a^2\right)^{-1}dx$, v will be an arctangent function which leads to an infinite loop instead of simplifying. Therefore, the new method here is to bring this function into the complex plane and solve using the Residue Theorem [6].

To use residue theorem, first convert the function into a complex integral so that $f(z)=\frac{\cos(z)}{z^2+a^2}$ where $z=x+iy$. $\cos(z)$ is substituted by e^{iz} because of what happens when our semi-circle's loop approaches a really high value. If it is just a real function cos(x), the graph is a wave between -1 and 1. However, when the semicircle approaches infinity (y approaches infinity), the complex function cos(z) approaches infinity as well because the complex function $\cos(z)=\frac{e^{iz}+e^{-iz}}{2}$, therefore $\cos(iy)=\frac{e^{y}+e^{-y}}{2}$ which approaches infinity as y approaches infinity. But if e^{iz} is used

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here, it takes advantage of the Euler's formula: $e^{iz} = \cos(z) + isin(z)$ as it keeps $\cos(z)$. What is also interesting is $e^{iz} = e^{ix}e^{-y},$ and e^{-y} approaches 0, that lead us to:

$$\oint \frac{e^{iz}}{z^2+a^2}\,dz = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+a^2}\,dx \tag{4}$$

To explain why the imaginary part $\oint \frac{\sin(x)}{(x^2+a^2)}\,dx$ is ignored, it can be proved through the rules of symmetry. In calculus, $sin(x)$ is an odd function, which means $-\sin(x) = \sin(-x)$. The denominator $x^2 + a^2$ is an even function. When integrating an odd function divided by an even function, the area under the negative side is equal to the area under the positive side, which perfectly adds up to 0 since they have opposite signs. Therefore, imaginary part of $\oint \frac{e^{iz}}{z^2+a^2}\,dz$ can be ignored.

While the Residue theorem provide a shortcut, the goal is eventually finding the definite integral for $\int_{-\infty}^{\infty} \int \frac{\cos(x)}{x^2+a^2}\,dx$, the bound is still real numbers and the answer is on the real number line. First, set a contour C, a semicircle including two parts, the path along the x-axis from -R to R (the real part), and a curved path in the upper half of the plane with radius R (the upper arc). When the limit as R approaches infinity, the real segment perfectly covers the bounds of the original integral. However, to apply the Residue Theorem, the person must prove that the contribution of the upper arc becomes negligible. If the integral along that arc does not vanish, the residue result would be invalid [7].

To analyze what happens on the arc, first define the path z as $z = Re^{i\theta}\ where\ 0 \le \theta \le \pi k.$ Insert this into the numerator to get $\left|e^{iz}\right| = \left|e^{i(Rcos(\theta)+iRsin(\theta)}\right| = \left|e^{i(Rcos(\theta)}\,e^{-Rsin(\theta)}\right|$. Since $\left|e^{ix}\right| = 1$ for any real x, the magnitude for $\left|e^{iz}\right|$ becomes $\left|e^{iz}\right| = e^{-Rsin(\theta)}$. Since θ is positive and $sin\,(\theta)$ is positive. The limit as R approaches infinity for the term$e^{-Rsin(\theta)}$decays to 0. This is what the imaginary part is different from the cos(x) function, and it proves the steps to show how the function vanishes on the upper bounds by extending e^{iz} using the Euler's identity.

This whole proof can be summarized by the Jordan's Lemma [8]. The original definition of Jordan's Lemma is when a function is defined as $f(z) = e^{imz}g(z)$ where $m > 0$, $g(z)$approaches zero as R approaches infinity. For this specific case, $m = 1$ (the coefficient of the exponent iz , and $g(z) = \frac{1}{z^2+a^2}$. This function obviously approaches zero as R approaches infinity. This is expressed as:

$$\lim_{R\to\infty} \left|\int_{\Gamma_R} \frac{e^{iz}}{z^2+a^2}\,dz\right| \le \int_0^{\pi} \frac{e^{-Rsin(\theta)}}{R^2-a^2}\,Rd\theta = 0 \tag{5}$$

where Γ_R is the upper arc. Since the integral along that arc Γ_R is equal to 0, it is confirmed that$\int_{-\infty}^{\infty} f(x)dx = \oint f(z)dz$ along the contour C. This whole process shows all the work that makes it valid to ignore the portion along the arc and confirm that the integral's value can be found by simply applying the residue theorem by adding up the residues and multiplying them by $2\pi i$.

The second step is to identify the poles of$f(z)$. Before trying to find the poles, it is important to know where the singularity is when the function diverges to infinity. In most cases, it occurs when the denominator is 0. In this case when$z^2 + a^2 = 0 \rightarrow (z - ia)(z + ia) = 0$. There are two

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isolated singularities at $z = \pm ia$. Since the contour mis restricted in the upper half of the plane, only $z = ia$ is inside the path. Since this singularity is a simple pole, the general equation for calculating residue looks like this:$Res\left(f, z_0\right) = \lim_{z \to z_0} \left(z - z_0\right) f\left(z\right)$. Substituting $z_0 = ia$:

$$Res\left(f, ia\right) = \lim_{z \to ia} \left(z - ia\right) \left(\frac{e^{iz}}{(z-ia)(z+ia)}\right) = \lim_{z \to ia} \frac{e^{iz}}{z+ia} \tag{6}$$

Substituting$z = ia,\ \text{Res}\left(f, ia\right) = \frac{e^{-a}}{2ia}$.

The Cauchy's Residue Theorem states that for a function analytic within and on a closed contour, the integral is:$\oint f\left(z\right) dz = 2\pi i \sum Res\left(f, z\right)$. Substituting into our result:$\oint \frac{e^{iz}}{z^2+a^2}\, dz = 2\pi i \left(\frac{e^{-a}}{2ia}\right) = \frac{\pi e^{-a}}{a}$. $\frac{\pi e^{-a}}{a}$ is the final answer to the integration after substituting e^{iz}, but the ultimate goal is to find the real part of that function by using the Euler's formula. In this case, since there is no imaginary part, the answer is still $I = \frac{\pi e^{-a}}{a}$.

3.2. Example II

The second case study will be analyzing the Dirichlet integral:

$$I = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx. \tag{7}$$

Unlike Example One where the singularities lie in the complex plain, this function has a pole at $z = 0$ after converting it to the complex plane. This makes it very different from the previous function because this singularity lies on the real axis, which cannot be simply driven over. To solve this problem, a new concept is introduced called the indented contour [9].

Define C as the indented contour that includes a semicircle Γ_R in the upper half that approaches infinity, and also two straight lines on the x-axis from$-R\ to\ -\epsilon$, and $\epsilon\ to\ R$where ϵ approaches 0 and R approaches infinity. Another tiny semicircle$\gamma_\epsilon$with radius ϵ that skips over the origin. This indented contour successfully included the real axis while detouring the singularity at $z = 0$.

Before moving to the actual calculation, first transition the function to a complex function$f(z)$, and substitute the numerator $sin(z)$ as e^{iz} as stated previously. Similar to$cos(z)$in the previous case,$sin(z)$is also a function that acts unstable as z approaches infinity. Therefore, it takes advantage of e-exponent e^{iz} 's growth and decay properties. As the upper half arc tends to infinity, e^{-y} tends to zero and satisfy the Jordan's Lemma, allowing the large arc to equate to zero.

To solve this problem, there is another new theory that needs to be introduced, The Half Residue Lemma [9]. In case I, the pole is covered by a full circle that is 360 degrees, this is why the factor is 2 π in the Residue Theorem. However, in case II, γ_ϵ is a semicircle that covers only half of the pole, and since the direction is clockwise (negative in complex analysis), the Residue Theorem here is changed to Half Residue:

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$$\int_{\gamma_\epsilon} f(z)dz = -\pi i \times Res(f,0) \tag{8}$$

After applying the indented contour, there is this function with no singularities inside. According to the Cauchy's Integral Theorem, if a function is analytic everywhere inside the contour, the total integral must equal 0: $\oint_C f(z)dz = 0$, i.e.,

$$\int_{Segments} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{\gamma_\epsilon} f(z)dz = 0 \tag{9}$$

First, focus on the large semicircle Γ_R. On this arc, the integral is equal to zero according to Jordan's Lemma explained in Case I. Because our function will contain e^{iz} again, the magnitude of e^{-y} tends to zero as the radius R approaches infinity. Therefore, it has been proven that the integral on the arc Γ_R equals zero, so this part can be ignored [10].

The next calculation is the small semicircle γ_ϵ where the pole is detoured at z=0. To calculate the residue here, use the Half-Residue formula introduced above because this arc only covers half the pole. The residue at $z = 0$ of $f(z) = \frac{e^{iz}}{z}$ is calculated by:

$$Res(f,0) = \lim_{z\to 0}(z)\left(\frac{e^{iz}}{z}\right) = e^{i0} = 1 \tag{10}$$

After plugging that into $-\pi i \times Res(f,0)$, the final residue for the semicircle $\gamma_\epsilon$is $-\pi i$. Now, the formula looks like this: $\int_{-\infty}^{\infty} f(x)dx - \pi i = 0,\ therefore \int_{-\infty}^{\infty} f(x)dx = \pi i.$ After getting this formula, apply the steps similar in the last case. Separate f(z) using the Euler's formula and it yield: $\int_{-\infty}^{\infty} \frac{\cos(x)+isin(x)}{x}\,dx = \pi i$. Since the general function is $\frac{\sin(x)}{x}$, look at the imaginary part:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi i \tag{11}$$

As introduced in case study I, do the same thing here but on the real part$\int_{-\infty}^{\infty} \frac{\cos(x)}{x}\,dx$. Since $g(x) = \frac{cos(x)}{x}$ is an odd function where$g(-x) = -g(x)$, the integral from the bound -R to R equals zero, therefore this part can be ignored and the final answer is $\int_{-\infty}^{\infty} \frac{sin(x)}{x}\,dx = \pi i$.

4. Conclusion

In conclusion, this paper effectively extends real calculus into the complex realm to solve for two special functions involved with trigonometry. By utilizing Jordan's Lemma, it proves that the as the radius approaches infinity, the large semicircle defined as Γ_R approaches zero, allowing the evaluation to go back to real calculus. By applying the indented contour, it effectively evaluated the Dirichlet's Integral through bypassing the singularities on the real axis. The substitution of e^{iz} using the Euler's formula significantly simplified the complexity of functions with trigonometric

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numerators, therefore making the integration easier. It shows how the imaginary number i acts as a bridge between many real-world problems.

However, the mathematical influence of the Residue Theorem is far beyond these two cases. Ever since the Residue Theorem was discovered, it helped solved many integrals that are thought to be unsolvable before, and it supported the development of many fields of science. In engineering, it allows engineers to predict the stability of architects such as bridges and skyscrapers. Engineers are also able to solve the Inverse Laplace Transforms, which is effective in signal processing. In Fluid Dynamics, it helped us calculate circulation. In quantum physics, it allows physicists to study the movement of particles using complex integrals.

References

[1] Ablowitz, M. J., & Fokas, A. S. (2021). Complex variables: Introduction and applications (3rd ed.). Cambridge University Press.
[2] Brown, J. W., & Churchill, R. V. (2013). Complex variables and applications (9th ed.). McGraw-Hill Education.
[3] Cain, G. (n.d.). Some applications of the residue theorem [Course supplement]. Georgia Institute of Technology. https: //cain.math.gatech.edu/notes/complex/resiapp.pdf
[4] Guo, Y., & Liu, J. (2024). The residue theorem approach to compute definite integrals. Highlights in Science, Engineering and Technology, 88, 469–474.
[5] Knill, O. (1996). The residue theorem and its applications [Lecture notes]. Department of Mathematics, California Institute of Technology. https: //people.math.harvard.edu/~knill/teaching/residues_1996/residue.pdf
[6] Lin, Z. (2021). On Cauchy's theorem and residue theorem of complex variable functions [Xuexi KeXi jifen ji liushu dingli de tihui]. Studies in College Mathematics, 24(1), 74–76.
[7] Rainwater, G. (2007). Residue theorems and their applications: Computing integrals once thought impossible to evaluate analytically [Bachelor's capstone project, California State University, Monterey Bay]. Digital Commons @ CSUMB. https: //digitalcommons.csumb.edu/caps_thes/361
[8] Zhao, Y. (2023). Residue theorem and its application. Highlights in Science, Engineering and Technology, 38, 290–299.
[9] Zhou, W., Liu, Y., & Song, T. (2022). Two types of infinite integrals handled by residue theorem. Journal of Inner Mongolia University (Natural Science Edition), 32(1), 56–60.
[10] Zhou, Y., & Huang, W. (2022). Further discussion on two types of infinite integral treated by residue theorem. College Physics, 41(9), 24–27.