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Mathematical Structures in Music: From Pythagorean Ratios to Group Theory

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Abstract. Mathematical structures have played an important role in the development of music theory, but the historical connection between numerical and structural approaches is not always presented systematically. This historical review examines how mathematical ideas are used to describe musical structures. It started with the development of the Pythagorean interpretation of harmony and numerical ratios, to the emergence of equal temperament and pitch-class thinking, and finally the approach of transformations and the application of group theory. In particular, it discussed the representation of the pitch classes using the additive group Z_{12}. This includes transposition, inversion, group actions and orbits. The review suggested that the mathematical approaches to music gradually evolved from measuring and describing individual musical intervals, to representing abstract musical objects, and finally analysing the transformations and structural relationships between them. And consequently, group theory creates a general mathematical framework for describing symmetry, equivalence, transformation, and invariance in musical structures.

Keywords: Mathematics, Music Theory, Pitch-Class Set Theorem, Group Theory

1. Introduction

People could recognise a melody even after it is transposed into another key or restated by another voice. Although the individual notes are changed, listeners continue to perceive the same musical idea. These ideas are not expressed through the isolated notes, but through how those notes relate to one another. Music provided a rich context for exploring numerical order, proportion, symmetry, and structural relationships, which maintained a close relationship with mathematics throughout the history. Over the past two millennia, mathematical ideas have not only contributed to the understanding of musical phenomena, but also influenced the development of music theory itself (e.g. [1]). Consequently, the study of mathematical structures in music has become a very important field connecting mathematics and the development of music theory.

Previous studies have examined this relationship from a variety of perspectives. The early investigations focused mainly on pitch systems, numerical proportions and harmonic ratios [2]. Long before mathematicians officially defined group theory, Renaissance and Baroque composers were already using transformational tricks (such as flipping or reversing melodies in canons), but perhaps without even knowing the math behind it. By the twentieth century, most topics of research expanded to group theory, including mutations, musical symmetry and modern algebraic approaches

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to composition [3, 4]. In particular, there is an increasing number of research describing musical structures through mathematical transformations in the twentieth century. This led to the application of concepts of group theory in the analysis of pitch, rhythm, and compositional techniques.

While those individual topics within mathematical music theory have been extensively studied, they are frequently presented as isolated developments. Historical studies explore how musical thought evolves, while mathematical frameworks focus on abstract modelling and structural analysis. As a result, relatively little attention has been paid to explain how these developments are historically connected. The idea of mathematical thinking gradually evolved from numerical descriptions of musical phenomena to structural descriptions. This is mainly based on symmetry and group transformations.

This review examines the historical evolution of mathematical structures in music and mainly focuses on the changing mathematical concepts used to represent those musical organisations. It begins with the Pythagorean explanation of harmony and numerical ratios. Then it explores the emergence of Pitch-Class thinking. It further discusses the rise of transformational approaches in twentieth century music and the application of group theory. Throughout this historical progression, some representative developments including Pythagorean framework, equal temperament and Pitch-Class Set theory are discussed, as they are the key milestones in the evolution of mathematical music theory. By reviewing these developments, this paper further argues that mathematical descriptions of music gradually evolved from numerical relationships to structural transformations.

2. The numerical origins of musical structure

The earliest systematic attempt to understand music through mathematics can be traced back to the Pythagorean school of ancient Greece. At that time musical harmony was first seen as a form of numerical order. Pythagoreans suggested that the pleasant musical intervals come from whole number ratios between string lengths, instead of a personal taste. Through experiments with the monochord, they found that cutting a string in half (2:1) creates an octave. Ratios like 3:2 and 4:3 produce a perfect fifth and a perfect fourth. These findings suggested that musical harmony depends on mathematical theories, but not just human perception [5, 6]. The Pythagorean approach did much more than finding the pleasing musical sounds. It created the first mathematical model of music. Instead of studying single notes, it focused on the relationships between them. That also associated with the Pythagorean view that "all is number" [7], since music is not seen just as art, but related to numbers in some ways.

However, the Pythagorean framework was mainly explaining consonance through those fixed numerical proportions, but not analysing music as a dynamic system. Musical practice gradually evolved from simple relationships to more complex compositional techniques. Consequently, this raised several new theoretical questions. Composers and theorists are concerned with why particular intervals sounded consonant, and with how those musical ideas can be preserved while undergoing various forms of transformation. Although numerical ratios are still indispensable for explaining tuning and harmonic relationships, they offered very limited tools to describe structural phenomena (such as transposition, inversion, and the organisation of larger musical forms) and became insufficient for the exploration of music theory [1, 8, 9].

These limitations did not diminish the importance of Pythagorean theory. Instead, they encouraged the development of mathematical explanations. Some later theorists sought for the mathematical model that can remains constant under the musical transformation, rather than the numerical relationships that can generate consonant sound. This is a fundamental change in the

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understanding of musical structure. The focus moved from the measurement of isolated intervals to the analysis of relationships between musical objects [9, 10].

3. The emergence of pitch-class thinking

As Western music became increasingly chromatic, the ratio-based tuning systems revealed more limitations. In Pythagorean tuning the twelve pure fifths do not exactly correspond to seven octaves. This creates a small difference known as the Pythagorean comma. It prevented the Pythagorean system from producing consonant intervals in every key (which creates restrictions to modulations). These difficulties encouraged the research for tuning systems that prioritised music flexibility over numerical ratios [10-12].

The common use of equal temperament represents another milestone. Equal temperament divides the octave into twelve equally spaced semitones, instead of treating each pitch as a unique frequency. This system no longer keeps the original idea of Pythagorean intervals. It establishes a unified space where every key has the same internal structure. Consequently, musical relationships became less dependent on absolute frequencies but more on relative positions within the octave [13, 14].

This perspective leads to the idea of pitch classes. It means the pitches that are one octave apart belong to the same equivalence class. This can be represented mathematically as the additive group $\mathbb{Z}_{12}$ [2], where each pitch class is represented by an integer modulo twelve. The octave is not just an interval, but an equivalence relation. This relation organises pitches into a finite mathematical structure. As a result, musical objects can be represented independently of their physical frequencies, which is more effective for analysing musical structures than numerical ratios [8, 15, 16].

4. The thinking of group theory in music

As pitches are represented as elements of an abstract space such as $\mathbb{Z}_{12}$, musical transformations could be treated as mathematical operations acting on these elements. This is a way to systematically describe the relationships between the musical objects. Therefore, mathematical music theory focuses more from identifying musical objects to analysing the transformations that preserve relationships of the structure between them [2, 17].

A simple example of these transformations is transposition. In the twelve-tone pitch system, each pitch class can be represented by an integer: $\mathbb{Z}_{12} = \{0,1,2,\ldots,11\}$, where each number corresponds to a pitch class modulo 12. Then, a transposition by n semitones can be expressed as the function: $T_n(x) = x + n\ (mod 12)$.

For example, a C major triad can be represented as: $X = \{0,4,7\}$.

Where 0, 4, and 7 represent C, E, and G respectively.

In addition, applying the transformation T_5 x gives: $T_5(x) = \{5,\ 9,\ 0\}$, which corresponds to an F major triad.

Although the specific pitches have changed, the interval structure of the chord remains the same. This mathematical operation showed that a musical object can change its position while keeping its internal relationships.

Inversion is another fundamental transformation in the twentieth-century. It reverses the direction of intervals around a reference point: $I(x) \equiv -x\ (mod 12)$.

For example, a pitch class 4 is transformed into: $I(4) = -4 \equiv 8\ (mod 12)$.

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Through these operations, musical transformations become mappings from one mathematical object to another. These mappings provide a precise language to describe the relationships that had previously been explained mainly through musical intuition [11, 16].

These twelve transpositions form a mathematical structure known as a group. Groups can be treated as a set satisfies the standard group axioms of closure, associativity, identity, and inverses.

All the twelve possible transpositions are: T_0, T_1, T_2, $\ldots$, T_{11}, they form the cyclic group: C_{12} since combining two transpositions produces another transposition: $T_a \circ T_b = T_{a+b}$, where the addition is performed modulo twelve.

The identity element is: T_0, this leaves every pitch unchanged, while each transformation has an inverse: $T_n^{-1} = T_{12-n}$ returning the musical object to its original position. Therefore, the mathematical properties required by group theory can be naturally emerge from the structure of musical transformations themselves [2, 4, 18].

Group theory further describes individual transformations in music. A group does not simply contain a collection of operations. It describes how these operations act upon specific musical objects.

This relationship can be formalised by the concept of group action: $G \times X \to X$, where G represents a group of transformations and X represents a set of musical objects. For example, the group G acts on a set of Pitch-Classes by transposition, generating all possible transpositions of a musical structure: $Orb(X) = \{gX \mid g \in G\}$.

This concept of an orbit explains why different musical fragments can be considered equivalent under the chosen group action, although they have different pitches [18-20]. They belong to the same structural class because one can be transformed into another. This is done through an element of the same group. This principle is fundamental to Pitch-Class Set Theory, where composers and theorists analysed collections of notes according to their transformational relationships, rather than their absolute pitch content [2, 4, 11].

Later theories on transformation further expanded this approach. In Neo-Riemannian theory [21], transformations such as P, L, and R describe relationships between chords through operations. It not only considered the traditional harmonic functions. These approaches suggest that the idea of group theory provides a general framework for understanding musical structure.

5. Conclusion

This review has examined the historical connection between numerical and structural approaches to describe musical structures. It started from the development of Pythagorean interpretation to the emergence of equal temperament, to pitch-class thinking, and finally to the transformational approaches and the application of group theory. It suggests that mathematical ideas used to describe musical structure gradually evolved from numerical relationships to structural transformations.

However, this review mainly focuses on Western music and on mathematical structures related to pitch and transformation. These models are not very representative for the perceptual, cultural, and emotional aspects of music. Future research could extend this historical perspective to non-Western musical systems and explore how computational and other mathematical approaches can be used to describe musical structures beyond.

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