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A Poset-Theoretic Approach to Reduced Cozero-Divisor Graphs of $\mathbb{Z}_n$ and Related Rings

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Abstract. Let R be a finite commutative ring with identity, and let $\Gamma'_r(R)$ denote its reduced cozero-divisor graph. We organize $\Gamma'_r(R)$ through the poset Prin*(R) of nonzero proper principal ideals and prove that adjacency is exactly incomparability in this poset. Consequently, cliques correspond to antichains, independent sets correspond to chains, and the chromatic number is determined by the width through Dilworth's theorem, in particular, every such graph is perfect. For the cyclic ring Z_n, the prime-power factorization of n identifies the full ideal lattice with a finite product of chains. A symmetric-chain decomposition of this product then shows that its width is the largest rank size, equivalently the central coefficient of its rank-generating polynomial. This yields explicit formulas for the number of vertices, clique number, chromatic number, and independence number of $\Gamma'_r(Z_n)$, together with specialized descriptions for prime, prime-power, and squarefree moduli. We also derive corresponding results for rings whose principal ideals form a chain and for finite Boolean rings. Finally, examples demonstrate both the computational strength and the reconstruction limits of the poset model: distinct rings may have isomorphic reduced graphs, whereas other small rings remain distinguishable. These results provide a unified order-theoretic framework for reduced cozero-divisor graphs and reduce several graph-invariant problems to finite-poset calculations.

Keywords: Reduced cozero-divisor graph, principal ideal, incomparability graph, cyclic ring, product of chains

1. Introduction

For a commutative ring R with identity, the cozero-divisor graph has vertex set consisting of the nonzero nonunits of R, with distinct vertices x and y adjacent precisely when $x \notin Ry$ and $y \notin Rx$. This graph was introduced by Afkhami and Khashyarmanesh to encode divisibility relations among ring elements in graph-theoretic form [1]. The reduced cozero-divisor graph identifies elements that generate the same principal ideal, thereby discarding multiplicity inside each principal-ideal class while retaining the relevant containment data. This construction was developed by Wilkens, Cain, and Mathewson [2], we denote the resulting graph by $\Gamma'_r(R)$. More broadly, algebraically defined graphs can translate algebraic symmetry into graph structure and suggest extremal questions. Cayley graphs and Ramsey-type questions provide a broad motivation for studying graphs generated by

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algebraic data, no Ramsey–Cayley assertion is used in the results below. The results established here rest on a single organizing observation: $\Gamma_r'(R)$ is the incomparability graph of the poset of nonzero proper principal ideals. For $R=\mathbb{Z}_n$, this model determines the number of vertices together with the clique, independence, and chromatic numbers from the prime exponents of n. We also treat the resulting special families, including chain rings and finite Boolean rings, and give examples that delimit the ring-theoretic information retained by the reduced graph.

Section 2 fixes the graph- and poset-theoretic terminology. Section 3 establishes the principal-ideal incomparability model, and Section 4 applies it to cyclic rings. Section 5 records examples and special families, while Section 6 examines reconstruction limits. We conclude in Section 7 with a summary and directions beyond the present scope.

2. Preliminaries

All rings are commutative with identity and $1\neq 0$. All graphs are simple and undirected. Finiteness is assumed whenever clique, independence, chromatic, width, or height maxima are invoked. For a finite graph G, write $\omega(G)$, $\alpha(G)$, and $\chi(G)$ for its clique, independence, and chromatic numbers. We use $\omega(K_0)=\alpha(K_0)=\chi(K_0)=0$.

For a ring R, set

$$W^*(R)=\{x\in R: x\neq 0,\ x\notin R^\times\}. \tag{1}$$

The *cozero-divisor* graph of R is the graph with vertex set $W^*(R)$ in which distinct vertices x and y are adjacent if and only if

$$x\notin Ry \qquad \text{and} \qquad y\notin Rx.$$

On $W^*(R)$ define an equivalence relation by

$$x\sim y \quad\Leftrightarrow\quad Rx=Ry. \tag{2}$$

The *reduced cozero-divisor graph* $\Gamma_r'(R)$ is the quotient graph whose vertices are the equivalence classes under $\sim$. Two distinct classes are adjacent precisely when $x\notin Ry$ and $y\notin Rx$ for representatives x and y, because this condition depends only on Rx and Ry, it is independent of the representatives chosen. We write

$$\mathrm{Prin}^*(R)=\{Rx: x\in W^*(R)\} \tag{3}$$

and order this set by inclusion. Thus $\mathrm{Prin}^*(R)$ is the poset of nonzero proper principal ideals represented by vertices of the cozero-divisor graph.

A *finite poset* is a finite set P equipped with a reflexive, antisymmetric, and transitive relation $\leq$. A *chain* is a subset of P whose elements are pairwise comparable, whereas an *antichain* is a subset whose distinct elements are pairwise incomparable. The *width* $\mathrm{width}(P)$ is the maximum cardinality of an antichain, and the *height*$\mathrm{height}(P)$ is the maximum cardinality of a chain. We set both parameters equal to 0 when $P=\varnothing$. These are the standard conventions and terminology used for finite posets, see, for example, [3, 4]. The order dual P^{op} of a poset P has the same underlying set as P, with

DOI: 10.54254/2753-8818/2026.PU36967

$$u \leq_{P^{op}} v \quad \Leftrightarrow \quad v \leq_P u.$$

The *incomparability graph* $\mathrm{Inc}(P)$ of a poset P has vertex set P, with two distinct elements adjacent exactly when they are incomparable in P. If $S \subseteq P$, the *induced subposet* on S is the poset whose order is the restriction of the order on P to S. A finite graph is *perfect* if every induced subgraph H satisfies $\chi(H) = \omega(H)$.

We use the following standard result without proof.

Theorem 2.1 (Dilworth's theorem [5]): For every finite poset P, the minimum number of chains whose union is P is equal to $\mathrm{width}(P)$.

3. The principal-ideal poset model

The defining membership conditions for adjacency can be translated into inclusion relations between principal ideals by the following elementary observation.

Lemma 3.1 (Membership and principal-ideal inclusion).

For any $x, y \in R$,

$$x \in Ry \quad \Leftrightarrow \quad Rx \subseteq Ry.$$

Proof.

Suppose first that $x \in Ry$. Then $x = ry$ for some $r \in R$. Every element of Rx therefore has the form

$$ax = a(ry) = (ar)y \in Ry,$$

so $Rx \subseteq Ry$. Conversely, if $Rx \subseteq Ry$, then $x = 1x \in Rx \subseteq Ry$. This proves both implications.

Theorem 3.2 (The principal-ideal incomparability model).

The map $[x] \mapsto Rx$ is a graph isomorphism

$$\Gamma'_r(R) \cong \mathrm{Inc}(\mathrm{Prin}^*(R), \subseteq). \tag{4}$$

Proof.

Define

$$\Phi : W^*(R)/\sim \to \mathrm{Prin}^*(R), \qquad \Phi([x]) = Rx.$$

If $[x] = [y]$, then $Rx = Ry$ by the definition of $\sim$, so Φ is well defined. It is surjective by the definition of $\mathrm{Prin}^*(R)$, and if $\Phi([x]) = \Phi([y])$, then $Rx = Ry$, whence $[x] = [y]$. Thus Φ is a bijection.

For two distinct classes $[x]$ and $[y]$, the membership lemma gives

$$[x] \sim_{\Gamma'_r(R)} [y] \quad \Leftrightarrow \quad x \notin Ry \text{ and } y \notin Rx$$

$$\Leftrightarrow \quad Rx \not\subseteq Ry \text{ and } Ry \not\subseteq Rx$$

$$\Leftrightarrow \quad \Phi([x]) \sim_{\mathrm{Inc}(\mathrm{Prin}^*(R))} \Phi([y]).$$

Hence Φ preserves and reflects adjacency and is a graph isomorphism.

DOI: 10.54254/2753-8818/2026.PU36967

Corollary 3.3 (Cliques, independent sets, antichains, and chains).

If $\mathrm{Prin}^*(R)$ is finite, then

$$\omega(\Gamma_r'(R))=\mathrm{width}(\mathrm{Prin}^*(R)), \qquad \alpha(\Gamma_r'(R))=\mathrm{height}(\mathrm{Prin}^*(R)). \tag{5}$$

Proof.

By Theorem 3.2, a set of vertices is a clique in $\Gamma_r'(R)$ exactly when the corresponding principal ideals are pairwise incomparable, that is, when they form an antichain in $\mathrm{Prin}^*(R)$. Taking maximum cardinalities gives $\omega(\Gamma_r'(R))=\mathrm{width}(\mathrm{Prin}^*(R))$.

Likewise, a set of vertices is independent exactly when every two of the corresponding ideals are comparable. Such a set is a chain, and every chain gives an independent set. Therefore $\alpha(\Gamma_r'(R))=\mathrm{height}(\mathrm{Prin}^*(R))$.

Corollary 3.4 (Coloring and perfectness).

If $\mathrm{Prin}^*(R)$ is finite, then

$$\chi(\Gamma_r'(R))=\omega(\Gamma_r'(R)), \tag{6}$$

and $\Gamma_r'(R)$ is perfect.

Proof.

Put $P=\mathrm{Prin}^*(R)$. Under the incomparability model, every color class is an independent set and hence a chain in P, conversely, every partition of P into chains gives a proper coloring of $\mathrm{Inc}(P)$. Thus $\chi(\Gamma_r'(R))$ is the minimum number of chains in a partition of P. Every chain partition is a chain cover. Conversely, given a chain cover, assign each element of P to one containing chain. The resulting nonempty classes form a chain partition with no more classes than the cover, since each class is contained in one of the covering chains. Hence the minimum chain-partition size equals the minimum chain-cover size. By Theorem 2.1, this common minimum is $\mathrm{width}(P)$, which equals $\omega(\Gamma_r'(R))$ by Corollary 3.3.

If $U\subseteq P$, the subgraph of $\mathrm{Inc}(P)$ induced by U is the incomparability graph of the induced subposet $P|_U$. Applying the same argument to $P|_U$ shows that every induced subgraph has equal clique and chromatic numbers. Hence $\Gamma_r'(R)$ is perfect.

The remainder of the paper therefore computes the relevant principal-ideal posets and reads the graph invariants from their chain and antichain structure.

4. Reduced cozero-divisor graphs of $\mathbb{Z}n$

Proposition 4.1 (Ideals and divisors).

Let $n\geq 2$. Every ideal of $\mathbb{Z}_n$ is uniquely of the form (d) for a positive divisor d of n, and

$$(d)\subseteq(e) \quad \Leftrightarrow \quad e|d.$$

Consequently, the vertices of $\Gamma_r'(\mathbb{Z}_n)$ are indexed by the proper divisors d with $1<d<n$.

Proof.

Let $\pi:\mathbb{Z}\to\mathbb{Z}_n$ be the quotient map. For every ideal I of $\mathbb{Z}_n$, the inverse image $\pi^{-1}(I)$ is an ideal of $\mathbb{Z}$ containing $n\mathbb{Z}$. Hence $\pi^{-1}(I)=d\mathbb{Z}$ for a unique positive divisor d of n, and the ideal correspondence gives $I=(d)$. This also proves uniqueness.

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If $e|d$, write $d=eq$. Then every multiple of d in $\mathbb{Z}_n$ is a multiple of e, so $(d)\subseteq(e)$. Conversely, if $(d)\subseteq(e)$, then $d\in(e)$ in $\mathbb{Z}_n$. Thus $d=eq+nt$ for some integers q,t. Since $e|n$, it follows that $e|d$. Therefore

$$(d)\subseteq(e) \quad \Leftrightarrow \quad e|d.$$

Finally, $d=1$ gives the whole ring and $d=n$ gives the zero ideal. Deleting these two endpoints leaves precisely the divisors $1<d<n$, which index the vertices of the reduced graph.

Proposition 4.2 (The exponent box).

Let

$$n=\prod_{i=1}^{k} p_i^{e_i},$$

where the p_i are distinct primes and the e_i are positive integers. For $C_{e_i}=\{0,1,\ldots,e_i\}$ with its usual order, equip $\prod_{i=1}^{k} C_{e_i}$ with the coordinatewise order. The map

$$\left(\prod_{i=1}^{k} p_i^{a_i}\right)\mapsto(e_1-a_1,\ldots,e_k-a_k)$$

is an order isomorphism from the ideal lattice of $\mathbb{Z}_n$ onto $\prod_{i=1}^{k} C_{e_i}$. Under this isomorphism, $\mathrm{Prin}^*(\mathbb{Z}_n)$ corresponds to the exponent box with its minimum and maximum removed. If $\rho(b_1,\ldots,b_k)=b_1+\cdots+b_k$, then the rank-generating polynomial of the full exponent box is

$$\sum_{(b_1,\ldots,b_k)\in\prod_{i=1}^{k} C_{e_i}} x^{\rho(b_1,\ldots,b_k)}=\prod_{i=1}^{k}(1+x+\cdots+x^{e_i}). \tag{7}$$

Proof.

By unique factorization, each positive divisor of n has a unique form $d=\prod_i p_i^{a_i}$ with $0\le a_i\le e_i$. If $c=\prod_i p_i^{c_i}$ is another divisor, then

$$(d)\subseteq(c) \quad \Leftrightarrow \quad c|d \quad \text{by Proposition 4.1}$$

$$\Leftrightarrow \quad c_i\le a_i \quad (1\le i\le k)$$

$$\Leftrightarrow \quad e_i-a_i\le e_i-c_i \quad (1\le i\le k).$$

Consequently the displayed map is a bijection that preserves and reflects the order. The zero ideal (n) maps to $(0,\ldots,0)$, while the whole ring (1) maps to $(e_1,\ldots,e_k)$. Hence deleting those two ideals corresponds exactly to deleting the minimum and maximum of the box.

Writing $b_i=e_i-a_i$, the contribution of the i th coordinate to the rank-generating function is $1+x+\cdots+x^{e_i}$. Since the coordinates are chosen independently and their ranks add,

$$\sum_{(b_1,\ldots,b_k)} x^{b_1+\cdots+b_k}=\prod_{i=1}^{k}(1+x+\cdots+x^{e_i}),$$

as required.

For a finite graded poset of rank N, a *symmetric chain decomposition* is a partition into saturated chains such that the ranks of the minimum and maximum elements of each chain sum to N.

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Products of chains and their symmetric chain decompositions are standard tools in Sperner theory, see, for example, [3, 4].

Lemma 4.3 (Symmetric chain decompositions).

Every finite product of finite chains admits a symmetric chain decomposition. Hence its width equals the size of a largest rank layer.

Proof.

Write $C_m=\{0,1,\ldots,m\}$, with the usual order and rank. We first decompose a product of two such chains. After swapping the factors if necessary, assume $a\le b$. For $0\le j\le a$, set

$$D_j=\{(j,t):0\le t\le b-j\}\cup\{(s,b-j):j+1\le s\le a\}, \tag{8}$$

ordered first by increasing t and then by increasing s. Given $(u,v)\in C_a\times C_b$, if $v\le b-u$, then (u,v) lies in the first part of D_u. If $v>b-u$, then $j=b-v$ satisfies $0\le j<u\le a$, and (u,v) lies in the second part of D_j. The two alternatives are disjoint and in each case determine j uniquely. Thus the D_j form a disjoint cover of $C_a\times C_b$.

Consecutive elements of D_j differ by increasing exactly one coordinate by one, so they are covers in the ambient product. Hence D_j is saturated. Its first element has rank j, and its last element has rank $a+b-j$ (also when the second part is empty). Therefore each D_j is symmetric in the rank- $a+b$ product.

Every finite chain is rank-isomorphic to some C_{a_i}. For $n\ge 1$ put

$$P_n=\prod_{i=1}^{n} C_{a_i},\qquad \rho_n(x_1,\ldots,x_n)=\sum_{i=1}^{n} x_i,\qquad N_n=\sum_{i=1}^{n} a_i.$$

We induct on n. For the base case, $P_1=C_{a_1}$ is itself one saturated symmetric chain. Suppose that P_n has a symmetric chain decomposition indexed as

$$S_n=\{S_\lambda:\lambda\in\Lambda_n\},\qquad S_\lambda=\{x_{\lambda,0}\lessdot\cdots\lessdot x_{\lambda,\ell_\lambda}\}. \tag{9}$$

Let $r_\lambda=\rho_n(x_{\lambda,0})$. Since S_λ is saturated, each step raises rank by one, and its symmetry therefore gives

$$\rho_n(x_{\lambda,t})=r_\lambda+t,\qquad 2r_\lambda+\ell_\lambda=N_n.$$

The decomposition of P_n gives the disjoint block decomposition

$$P_{n+1}=P_n\times C_{a_{n+1}}=\bigsqcup_{\lambda\in\Lambda_n}(S_\lambda\times C_{a_{n+1}}). \tag{10}$$

On the individual block belonging to S_λ, define the relative-rank map

$$\Phi_\lambda:S_\lambda\times C_{a_{n+1}}\to C_{\ell_\lambda}\times C_{a_{n+1}}$$

$$\Phi_\lambda(x_{\lambda,t},s)=(t,s).$$

This is an order isomorphism, and the ambient rank is the grid rank shifted by r_λ :

DOI: 10.54254/2753-8818/2026.PU36967

$$\rho_{n+1}(x_{\lambda,t},s)=r_\lambda+t+s.$$

Apply the two-factor decomposition above to $C_{\ell_\lambda}\times C_{a_{n+1}}$ and pull every resulting chain back through Φ_λ^{-1}. A cover in a grid chain has one of the forms

$$(t,s)\lessdot(t+1,s)\quad\text{or}\quad(t,s)\lessdot(t,s+1).$$

Its inverse image is, respectively,

$$(x_{\lambda,t},s)\lessdot(x_{\lambda,t+1},s)\quad\text{or}\quad(x_{\lambda,t},s)\lessdot(x_{\lambda,t},s+1),$$

and these are covers in the ambient poset P_{n+1}. Conversely, Φ_λ sends each of these displayed ambient covers to the corresponding grid cover. Thus every pulled-back chain is saturated in P_{n+1}, not merely in its block. Within a block, bijectivity of Φ_λ makes the pulled-back chains a disjoint cover, the disjoint block decomposition above then gives disjoint coverage of all of P_{n+1}.

If a grid chain has endpoint ranks q and $\ell_\lambda+a_{n+1}-q$, its pulled-back endpoint ranks sum to

$$(r_\lambda+q)+\left(r_\lambda+\ell_\lambda+a_{n+1}-q\right)=2r_\lambda+\ell_\lambda+a_{n+1}=N_n+a_{n+1}=N_{n+1}.$$

Hence all the pulled-back chains are symmetric in P_{n+1}, completing the induction.

Finally, let the total rank be N. A symmetric saturated chain begins at some rank q and ends at ran $N-q$, so it contains exactly one element of rank $\lfloor N/2\rfloor$. Any antichain meets each chain at most once. The rank- $\lfloor N/2\rfloor$ layer is itself an antichain and meets every chain once, so its size equals the number of chains and the width. Since every rank layer is an antichain, this central layer is a largest rank layer.

Theorem 4.4 (Graph invariants for composite moduli).

Let $n=\prod_{i=1}^{k}p_i^{e_i}$ be composite, put $E=\sum_{i=1}^{k}e_i$, and define

$$F_n(x)=\prod_{i=1}^{k}(1+x+\cdots+x^{e_i}). \tag{11}$$

Then the following identities hold:

$$|V(\Gamma_r'(\mathbb{Z}_n))|=\prod_{i=1}^{k}(e_i+1)-2, \tag{12}$$

$$\omega(\Gamma_r'(\mathbb{Z}_n))=\chi(\Gamma_r'(\mathbb{Z}_n))=[x^{\lfloor E/2\rfloor}]F_n(x), \tag{13}$$

$$\alpha(\Gamma_r'(\mathbb{Z}_n))=E-1. \tag{14}$$

Proof.

The full exponent box contains $\prod_i(e_i+1)$ elements. By Proposition 4.2, its minimum and maximum correspond to the zero ideal and the whole ring, so deleting them gives

$$|V(\Gamma_r'(\mathbb{Z}_n))|=\prod_i(e_i+1)-2.$$

DOI: 10.54254/2753-8818/2026.PU36967

The box has total rank E. Because n is composite, $E \geq 2$, and the central rank $\lfloor E/2 \rfloor$ lies between 1 and $E-1$. It is therefore unchanged when the two endpoints are removed. By Lemma 4.3, the width of the full box is the central coefficient $[x^{\lfloor E/2 \rfloor}]F_n(x)$. Every antichain in the proper part is an antichain in the full box, while the central rank is an antichain contained in the proper part. Thus removing the endpoints does not change the width. Now Corollaries 3.3 and 3.4 give

$$\omega(\Gamma_r'(\mathbb{Z}_n)) = \chi(\Gamma_r'(\mathbb{Z}_n)) = [x^{\lfloor E/2 \rfloor}]F_n(x).$$

Only the ranks $1,\ldots,E-1$ occur in the proper part, and a chain has at most one element of each rank. Its length is therefore at most $E-1$. On the other hand, incrementing one exponent coordinate at a time produces a saturated chain from the minimum to the maximum of the full box, after the endpoints are removed, it contains exactly one element in every rank $1,\ldots,E-1$. The height is consequently $E-1$, and Corollary 3.3 yields $\alpha(\Gamma_r'(\mathbb{Z}_n)) = E-1$.

Corollary 4.5 (Squarefree moduli).

Let $n = p_1 \cdots p_k$ be a product of distinct primes, where $k \geq 2$. Then

$$|V(\Gamma_r'(\mathbb{Z}_n))| = 2^k - 2, \qquad \omega(\Gamma_r'(\mathbb{Z}_n)) = \chi(\Gamma_r'(\mathbb{Z}_n)) = \binom{k}{\lfloor k/2 \rfloor}, \qquad \alpha(\Gamma_r'(\mathbb{Z}_n)) = k-1. \tag{15}$$

Proof.

Here $e_i = 1$ for every i, so

$$\prod_{i=1}^{k}(e_i+1) = 2^k, \qquad E = k, \qquad F_n(x) = (1+x)^k.$$

Substituting these expressions into Theorem 4.4 gives the vertex count and the independence number. The central coefficient of $(1+x)^k$ is $\binom{k}{\lfloor k/2 \rfloor}$, which gives both the clique and chromatic numbers.

Corollary 4.6 (Prime and prime-power moduli).

Let p be prime. Then

$$\Gamma_r'(\mathbb{Z}_p) = K_0, \qquad \Gamma_r'(\mathbb{Z}_{p^e}) \cong \overline{K}_{e-1} \quad (e \geq 2). \tag{16}$$

Proof.

The only ideals of $\mathbb{Z}_p$ are the zero ideal and the whole ring. Hence there is no vertex and $\Gamma_r'(\mathbb{Z}_p) = K_0$.

Let $e \geq 2$. By Proposition 4.1, the nonzero proper ideals are

$$\{(p^j) : 1 \leq j \leq e-1\}.$$

They are ordered by $(p^a) \subseteq (p^b)$ exactly when $a \geq b$, so they form a reverse-exponent chain of $e-1$ elements (a singleton when $e=2$). Its incomparability graph has no edges, so the principal-ideal model gives $\Gamma_r'(\mathbb{Z}_{p^e}) \cong \overline{K}_{e-1}$.

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5. Examples and special families

For the examples below, if $d=\prod_i p_i^{a_i}$, write $r(d)=\sum_i a_i$ and let $L_j=\{d:1<d<n,\ r(d)=j\}$. Thus the L_j are the proper divisor rank layers. Reversing the coordinates as in Proposition 4.2 reverses the rank numbers but leaves their sizes unchanged.

Example 5.1 (The ring $\mathbb{Z}_8$).

Here $8=2^3$. The vertices and proper rank layers are

$$V=\{2,4\},\qquad L_1=\{2\},\qquad L_2=\{4\}.$$

The full rank polynomial is

$$F_8(x)=1+x+x^2+x^3.$$

A representative maximum clique is $\{2\}$, and a maximum chain is $2 \mid 4$. Consequently,

$$(|V|,\omega,\alpha,\chi)=(2,1,2,1).$$

Example 5.2 (The ring $\mathbb{Z}_{12}$).

Here $12=2^2\cdot 3$. The vertices and proper rank layers are

$$V=\{2,3,4,6\},\qquad L_1=\{2,3\},\qquad L_2=\{4,6\}.$$

The full rank polynomial is

$$F_{12}(x)=(1+x+x^2)(1+x)=1+2x+2x^2+x^3.$$

A representative maximum clique is $\{2,3\}$, and a maximum chain is $2 \mid 4$. Consequently,

$$(|V|,\omega,\alpha,\chi)=(4,2,2,2).$$

Example 5.3 (The ring $\mathbb{Z}_{30}$).

Here $30 = 2 \cdot 3 \cdot 5$. The vertices and proper rank layers are

$$V=\{2,3,5,6,10,15\},\qquad L_1=\{2,3,5\},\qquad L_2=\{6,10,15\}.$$

The full rank polynomial is

$$F_{30}(x)=(1+x)^3=1+3x+3x^2+x^3.$$

A representative maximum clique is $\{2,3,5\}$, and a maximum chain is $2 \mid 6$. Consequently,

$$(|V|,\omega,\alpha,\chi)=(6,3,2,3).$$

Example 5.4 (The ring $\mathbb{Z}_{72}$).

Here $72=2^3\cdot 3^2$, and the ten vertices are

$$V=\{2,3,4,6,8,9,12,18,24,36\}.$$

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Rather than drawing the relatively dense incomparability graph, we use the exponent-box order: represent $d=2^a3^b$ by (a,b) in $C_3\times C_2$, remove the two endpoints, and join precisely the incomparable pairs. The proper rank layers are

$$L_1=\{2,3\},\quad L_2=\{4,6,9\}$$

$$L_3=\{8,12,18\},\quad L_4=\{24,36\}.$$

The full rank polynomial is

$$F_{72}(x)=(1+x+x^2+x^3)(1+x+x^2)=1+2x+3x^2+3x^3+2x^4+x^5.$$

A maximum clique is $\{4,6,9\}$, and $2\mid 4\mid 8\mid 24$ is a maximum chain. Hence

$$(|V|,\omega,\alpha,\chi)=(10,3,4,3).$$

Proposition 5.5 (Principal-ideal chains).
If $\mathrm{Prin}^*(R)$ is a finite chain with t elements, then

$$\Gamma_r'(R)\cong\overline{K}_t.$$

Thus, for $t>0$, $\omega=\chi=1$ and $\alpha=t$, for $t=0$, all three invariants are zero.
Proof.
Any two elements of the chain $\mathrm{Prin}^*(R)$ are comparable. Its incomparability graph therefore has t vertices and no edges, and Theorem 3.2 gives $\Gamma_r'(R)\cong\overline{K}_t$.

If $t>0$, a nonempty edgeless graph has clique and chromatic numbers equal to 1, while all t vertices form an independent set. If $t=0$, the graph is K_0, and all three invariants are zero by the conventions fixed in Section 2.

Proposition 5.6 (Finite Boolean rings).
Every finite Boolean ring is isomorphic to F_2^k for some $k\geq 1$. Its full principal-ideal poset is the Boolean lattice B_k, and $\mathrm{Prin}^*(R)$ is obtained from B_k by deleting its minimum and maximum. For $k\geq 2$,

$$|V|=2^k-2,\qquad \omega=\chi=\binom{k}{\lfloor k/2\rfloor},\qquad \alpha=k-1. \tag{17}$$

For $k=1$, one has $\Gamma_r'(R)=K_0$.
Proof.
By the standard structure theorem for finite Boolean rings, a finite Boolean ring is a finite product of fields in which every element is idempotent [6]. In each field the identity $a^2=a$ gives $a(a-1)=0$, so the field has only the elements 0 and 1. Hence $R\cong F_2^k$ for some $k\geq 1$.

For $S\subseteq[k]$, let $e_S\in F_2^k$ be the vector whose i th coordinate is 1 exactly when $i\in S$. The principal ideal Re_S consists precisely of the vectors supported in S. Conversely, let I be an ideal and put

$$S=\{i:\mathrm{some}\,x\in I\,\mathrm{has}\,x_i=1\}.$$

DOI: 10.54254/2753-8818/2026.PU36967

If $i \in S$, multiplication by the i th coordinate idempotent shows that this idempotent belongs to I. Their sum e_S therefore belongs to I, so $\mathrm{Re}_S \subseteq I$. Every vector in I is supported in S by the definition of S, giving the reverse inclusion. Thus $I = \mathrm{Re}_S$, and

$$\mathrm{Re}_S \subseteq \mathrm{Re}_T \quad \Leftrightarrow \quad S \subseteq T.$$

The full principal-ideal poset is consequently B_k, and $\mathrm{Prin}^*(R)$ is obtained by deleting $\varnothing$ and $[k]$.

For $k \geq 2$, this leaves $2^k - 2$ vertices. Since $B_k \cong C_1^k$, Lemma 4.3 shows that the width of B_k is the size of its central rank, namely $\binom{k}{\lfloor k/2 \rfloor}$. This rank is internal because $1 \leq \lfloor k/2 \rfloor \leq k-1$, and its subsets form an antichain in the proper part, so that part has width at least $\binom{k}{\lfloor k/2 \rfloor}$. Conversely, every antichain in the proper part remains an antichain in B_k, whose width is $\binom{k}{\lfloor k/2 \rfloor}$, giving the matching upper bound. Hence endpoint deletion preserves the width, and Theorem 3.2, Corollary 3.3, and Corollary 3.4 give

$$\omega = \chi = \binom{k}{\lfloor k/2 \rfloor}.$$

A strict chain of nonempty proper subsets uses at most the sizes $1, \ldots, k-1$. Conversely, the subsets $S_j = \{1, \ldots, j\}$ for $1 \leq j \leq k-1$ form a chain of $k-1$ elements. Hence $\alpha = k-1$. When $k = 1$, deleting the two elements of B_1 leaves the empty poset, so $\Gamma_r'(R) = K_0$.

6. Reconstruction limits and ring-level examples

Proposition 6.1 (Dual invariance).

For every poset P,

$$\mathrm{Inc}(P) = \mathrm{Inc}(P^{\mathrm{op}})$$

on the same vertex set.

Proof.

Let u and v be distinct elements of P. They are comparable in P if and only if

$$u \leq_P v \quad \text{or} \quad v \leq_P u.$$

By the definition of the order dual, reversing the order merely exchanges the two alternatives in this disjunction. Thus u and v are comparable in P if and only if they are comparable in P^{op}. Taking the negation shows that the incomparable unordered pairs, and hence the edge sets of the two incomparability graphs, are identical.

Thus an incomparability graph does not determine the orientation of comparable pairs: reversing every comparison leaves the graph unchanged, so the graph alone cannot specify which of two comparable elements lies below the other.

Example 6.2 (Fields).

If F is a field, every nonzero element of F is a unit. Hence $W^*(F) = \varnothing$ and

$$\Gamma_r'(F) = K_0.$$

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Thus every field has the same empty reduced graph.

Example 6.3 (Two small rings).

By Proposition 5.6, the two nonzero proper principal ideals of F_2^2 are generated by $(1,0)$ and $(0,1)$. They are incomparable, and therefore

$$\Gamma_r'(F_2^2) \cong K_2.$$

By contrast, $F_2[x]/(x^2)$ has just one nonzero proper principal ideal, namely (x), so

$$\Gamma_r'\left(F_2[x]/(x^2)\right) \cong K_1.$$

In particular, the reduced graph distinguishes these two rings.

Example 6.4 (The triangular prism).

The squarefree divisor model underlying Corollary 4.5 and the Boolean-lattice description in Proposition 5.6 show that, for $\mathbb{Z}_{30}$ and F_2^3, respectively, the nonzero proper principal ideals are indexed by the nonempty proper subsets of a three-element set. In the corresponding incomparability graph, the three singleton subsets form one triangle, the three two-element subsets form another, and each singleton is joined to its complementary two-element subset. Thus

$$\Gamma_r'(\mathbb{Z}_{30}) \cong \Gamma_r'(F_2^3) \cong \text{the triangular-prism graph}.$$

The two rings are not isomorphic, since their characteristics are 30 and 2, respectively.

These examples show that the reduced graph can sometimes distinguish rings, but it does not determine the ring isomorphism type.

7. Conclusion

This paper has developed a poset-theoretic framework for reduced cozero-divisor graphs of finite commutative rings. The central point is that quotienting nonzero nonunits by equality of generated principal ideals does more than reduce the vertex set: it reveals the correct order structure. Under the natural identification of a vertex class with its principal ideal, two vertices are adjacent precisely when the associated ideals are incomparable. Thus, the reduced graph is the incomparability graph of the poset Prin*(R). This identification simultaneously explains the clique, independence, and coloring parameters. Maximum cliques are maximum antichains, maximum independent sets are maximum chains, and Dilworth's theorem gives equality of the chromatic and clique numbers. Applying the same argument to induced subposets also proves perfectness.

For cyclic rings, the method converts ring-theoretic data into explicit combinatorics. The exponents in the prime-power factorization of n identify the principal-ideal lattice of $\mathbb{Z}_n$, after reversing the divisor exponents, with a finite product of chains. Removing its two extremal elements produces the vertex poset. The constructive symmetric-chain decomposition proved here shows that the width of the full box is its largest rank size, so the central coefficient of the rank-generating polynomial introduced in Section 4 determines both the clique and chromatic numbers. The same exponent data give the vertex count and the independence number. The prime, prime-power, and squarefree cases follow transparently, and the Boolean-ring and principal-ideal-chain results show that the framework is not confined to cyclic rings.

The examples also clarify the limits of the invariant. The reduced graph remembers comparability only up to orientation and discards multiplicities inside principal-ideal classes, consequently, it

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cannot generally reconstruct the ring. This loss is illustrated by nonisomorphic rings with the same reduced graph, while contrasting examples show that the graph can still distinguish some rings. A natural next step is to characterize which finite principal-ideal posets arise from broader classes of rings and to determine which graph invariants survive products, local decompositions, or other ring constructions. Further work may also study metric parameters, automorphism groups, forbidden induced subgraphs, and reconstruction under additional algebraic hypotheses. Within the present scope, the main conclusion is that the principal-ideal poset is the effective intermediary: once it is identified, structural questions about the reduced graph become tractable problems in finite order theory.

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