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Group Actions with Two Faces: From Cayley Embedding to Conjugacy Decomposition and Structural Restrictions on Finite Groups

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Abstract. Since Galois, abstract group theory has occupied a central position in modern mathematics. The Orbit-Stabilizer Theorem provides the fundamental counting relation for group actions, balancing orbit size against the symmetry that fixes a point. This paper offers a self-contained exposition of the duality between permutation representations and internal structure in finite groups. Historically, the study of group actions evolved from Cayley's external embedding of every group into a symmetric group to the internal dissection via conjugation, where orbits become conjugacy classes and stabilizers become centralizers. This paper systematically develops both perspectives: the left regular action yields Cayley's theorem, while the conjugation action gives rise to the Class Equation. As a central application, this paper prove that every non-trivial finite p-group possesses a non-trivial centre. By contrasting two actions, this paper reveals how the same algebraic theorem, viewed through different lenses, simultaneously delivers an external realisation and an internal structural constraint, illustrating the interpretive richness of group theory.

Keywords: Group actions, Orbit-Stabilizer Theorem, Cayley's Theorem, Conjugation action, Finite p-group

1. Introduction

Abstract group theory turned into something central after Galois. Groups can act on sets by permuting their elements and that is where a lot of the usefulness comes from. The orbit stabilizer theorem gives the basic count for this. It says the orbit of a point has size equal to the group order divided by the stabilizer order. There is this balance between how big the orbit gets and how much symmetry stays fixed at that point. This simple equation captures a fundamental trade-off between orbit size and the amount of symmetry fixing the point [1].

Historically, the study of group actions evolved along two distinct but intersecting trajectories. The earliest systematic use, epitomized by Cayley's celebrated theorem in 1854, treated groups primarily as external objects of permutations. Cayley's insight that every abstract group can be realized as a subgroup of a symmetric group. This established the "permutation representation" paradigm, which dominated much of 19th-century group theory [2]. However, as the discipline matured, mathematicians recognized that placing a group in a concrete permutation context

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sometimes obscured its internal algebraic intricacies. This realization prompted a conceptual shift: instead of letting the group act on arbitrary external sets, one could let it act upon itselfto reveal its intrinsic architecture. The structure of conjugacy classes has emerged as a powerful invariant and has more recently been connected to the classification of finite simple groups. The introduction of the conjugacy action marked a pivotal turn in this direction. By examining how elements are partitioned into conjugacy classes, early finite group theorists discovered the centralizers carry precise information about commutativity and normal subgroups. This internal perspective culminated in the derivation of the Class Equation, which quickly became a cornerstone of the field. Landmark results, such as the Feit–Thompson odd-order theorem [3], underscore how class sizes can force solvability or simplicity, while the celebrated work of Burnside on p -complements and Glauberman's ZJ -theorem reveal the subtle interplay between centralizers and global structure [4-6]. Recent progress on the "extreme" cases (groups with few conjugacy classes versus those with highly symmetric Cayley embeddings) suggests that the tension between these two faces is not merely descriptive but prescriptive: each imposes formidable restrictions on the other.

This paper offers a self-contained exposition of the duality between permutation representations and internal structure in finite groups, as revealed by the Orbit-Stabilizer Theorem through left multiplication and conjugation. The remainder of this paper is structured as follows. Section 2 reviews the basic definitions of group actions and the Orbit-Stabilizer Theorem. Section 3 looks at the regular action and Cayley. Section 4 works out the class equation from the conjugation action and applies this to prove the p -group center theorem, then try to contrast the two actions. Finally, section 5 concludes the paper.

2. Preliminaries on group actions

This section talks about group actions and how orbits and stabilizers work. It gives the proof for the Orbit-Stabilizer Theorem too. That sets up the left regular action and the conjugation action for later on.

2.1. Basic definitions of group actions

The essence of a group action is to "realize" each abstract element of a group as a permutation on some set. Different actions can show different sides of the group. It is not always the same realization.

2.1.1. Definition of a group

Consider a non-empty set L and a binary operation on it, denoted by

$$L \times L \to L. \tag{1}$$

Under this operation, the pair (d, f) corresponds to the element $d \cdot f$. Such a set together with this operation forms a group, which satisfying the three properties:

a. Associativity: For all $d, f, j \in L, (d \cdot f) \cdot j = d \cdot (f \cdot j)$.

b. Identity: There exists an element $e \in L$ such that for every $d \in L, e \cdot d = d \cdot e = d$.This element is unique and it is the identity of L.

c. Inverse: For each $d \in L$, there is an element $b \in L$ such that $d \cdot b = b \cdot d = e$. This element is unique and is denoted by a^{-1}.

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2.1.2. Definition of a left group action

Define a group L and a non-empty set F. L acts on the left on F if there is a map

$$L \times F \to F,\ (r, f) \mapsto r \cdot f \tag{2}$$

satisfying the following two axioms:
Identity: for all $f \in F, e \cdot f = f$, where e is the identity element of L.
Compatibility: for all $l_1, l_2 \in L$ and $f \in F$, $(l_1 l_2) \cdot f = l_1 \cdot (l_2 \cdot f)$.
These two axioms ensure that each $l \in L$ induces a bijection on F, and that the map

$$\varphi :\ L \to \mathrm{Sym}\,(F),\ \varphi\,(l)\,(f) = l \cdot f \tag{3}$$

is a group homomorphism. This is the one-to-one correspondence between group actions and homomorphisms to symmetric groups, which is the starting point for Cayley's theorem [7, 8].

Remark: In the subsequent discussion, this paper focuses mainly on the case where L acts on its carrier set (i.e., $\mathrm{F} = \mathrm{L}$). Although the set is the same, different action rules (left multiplication or conjugation) will yield completely different structural information [9].

2.1.3. Definition of orbit

The action of L on F induces an equivalence relation, where two points are deemed equivalent whenever some group element carries one to the other. The resulting equivalence classes are precisely the orbits.

Orbit: For a fixed $f \in F$, its corresponding orbit is defined as the subset

$$O_x = \{h \cdot f \mid h \in L\} \subseteq F. \tag{4}$$

Equivalently, this orbit collects all possible images of f under the entire group L. It is well known that these disjoint orbits completely decompose the whole set F.

2.1.4. Definition of stabilizers

For $f \in F$, its stabilizer is defined as the subset

$$\mathrm{Stab}\,(f) = \{h \in L \mid h \cdot f = f\} \subseteq L \tag{5}$$

One readily checks that $\mathrm{Stab}\,(x)$ forms a subgroup of L. This subgroup indicates which group elements leave f unchanged.

There is a trade-off between the orbit size and the stabilizer size, which lies at the heart of the following subsection.

2.2. The orbit-stabilizer theorem

The Orbit-Stabilizer Theorem provides a quantitative bridge between the "set-level" and the "group-level", and it is the cornerstone of all subsequent derivations in this paper [10].

Suppose L is finite and acts on F. For any $f \in F$, then theorem asserts that the cardinality of the orbit $L \cdot x$ coincides with the index of the stabilizer L_x in L :

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$$|L \cdot x| = [L : L_x] \tag{6}$$

In particular,

$$|L| = |L_x| \cdot |L \cdot x| \tag{7}$$

Proof. The argument proceeds by constructing a bijection between the left coset space $L/L_x = \{mL_x : m \in L\}$ and the orbit $L \cdot x$. Define

$$\varphi : L/L_x \longrightarrow L \cdot x,\ \varphi(mL_x) = m \cdot x \tag{8}$$

First check that φ is well-defined. Suppose $m_1 L_x = m_2 L_x$. Then $m_1^{-1} m_2 \in L_x$. By definition of the stabiliser, this means

$$\left(m_1^{-1} m_2\right) \cdot x = x. \tag{9}$$

Applying the group action axioms,

$$m_1^{-1} \cdot (m_2 \cdot x) = x. \tag{10}$$

Now apply m_1 to both sides of (10), obtaining

$$m_2 \cdot x = m_1 \cdot \left(m_1^{-1} \cdot (m_2 \cdot x)\right) = m_1 \cdot x. \tag{11}$$

Hence $\varphi(m_1 L_x) = \varphi(m_2 L_x)$, proving that φ is well-defined.

Next, show that φ is injective.

If $\varphi(m_1 L_x) = \varphi(m_2 L_x)$, then $m_1 \cdot x = m_2 \cdot x$. Left multiplication by m_1^{-1} yields

$$x = m_1^{-1} \cdot (m_2 \cdot x) = \left(m_1^{-1} m_2\right) \cdot x. \tag{12}$$

Thus $m_1^{-1} m_2 \in L_x$, which implies that $m_1 L_x = m_2 L_x$. Therefore, φ is injective.

Surjectivity is immediate from the definition of the orbit: any element $L \cdot x$ has the form $m \cdot x = \varphi(mL_x)$ for some $m \in L$.

Therefore φ is a bijection between L/L_x and $L \cdot x$. Since L is finite, the index $[L : L_x]$ equals $|L| / |L_x|$; consequently,

$$|L \cdot x| = [L : L_x] \text{ and } |L| = |L_x| \cdot |L \cdot x|. \tag{13}$$

The proof is now complete.

In the following sections, when applied to different actions, it yields two completely different types of conclusions: an "external embedding" and an "internal classification" of the group [11, 12].

3. The left regular action and Cayley's embedding

This section will first examine the simplest and straightforward way for a group to act upon itself, namely by left multiplication. The salient feature of this action is its strong transitivity, it glues the

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entire group into a single orbit, thereby embedding the abstract group faithfully into a symmetric group [2].

3.1. Construction of the left regular action

Define a group L. Take the set $F = L$ (viewing the group merely as a set). Define the action

$$L \times L \to L, \ \ (m, x) \mapsto m \cdot x = mx \tag{14}$$

where the multiplication on the right is the group operation. This is the left regular action.

Validate the action properties:

a. Identity: $e \cdot x = ex = x$;

b. Compatibility $(m_1 m_2) \cdot x = (m_1 m_2)x = m_1 (m_2 x) = m_1 \cdot (m_2 \cdot x)$.

Thus, this is a legitimate group action. Here, each group element h corresponds to the permutation $L_h : x \mapsto hx$ on the set G, which is simply "left translation" [1, 2].

3.2. Transitivity and trivial stabilizers

Then, analyzing the orbits and stabilizers under this action, as these will be the key to applying the Orbit-Stabilizer Theorem.

3.2.1. Uniqueness: the action is transitive

For any two elements $x, y \in G$, can always find $h = yx^{-1} \in G$ such that

$$h \cdot x = \left(yx^{-1}\right)x = y. \tag{15}$$

Since any two points can be transformed into each other by some group element, the entire set G itself is a single orbit

$$O_x = G \tag{16}$$

for every $x \in G$ [4].

3.2.2. The stabilizer degenerates to the trivial subgroup

For any $x \in L$,

$$\text{Stab}\,(x) = \{m \in L \mid m \cdot x = x\} = \{m \in L \mid mx = x\}. \tag{17}$$

By right cancellation in a group, $mx = x$ implies $m = e$. Hence

$$\text{Stab}\,(x) = \{e\}. \tag{18}$$

Under this action, the symmetry at every point is minimal, no non-identity element fixes any point [11].

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3.3. Derivation of Cayley's theorem

Now, apply the Orbit-Stabilizer Theorem to the left regular action.

Taking any $x \in L$,

$$|O_x| = |L|, |Stab(x)| = 1. \tag{19}$$

Substituting into the theorem $|O_x| \cdot |Stab(x)| = |L|$ gives $|L| \times 1 = |L|$, an identity that confirms the consistency of the computation [12].

Adopting a homomorphism viewpoint reveals a deeper outcome. The action induces a mapping

$$\varphi : L \to \mathrm{Sym}(L),\ \varphi(h) = L_h, \tag{20}$$

where $\mathrm{Sym}(L)$ is the permutation group on L.

Compute its kernel:

$$\ker \varphi = \{h \in L \mid L_h = \mathrm{id}\} = \{\mathrm{h} \in L \mid \forall x \in L,\ hx = x\}. \tag{21}$$

Taking $x = e,\ he = e \Rightarrow h = e$.

Hence the kernel contains only the identity element,

$$i.e.\ ker\varphi = \{e\}. \tag{22}$$

Invoking the First Isomorphism Theorem, it follows that $L \cong \varphi(L) \leq \mathrm{Sym}(L)$. For $|L| = n$, since $\mathrm{Sym}(L) \cong S_n$, obtain the classical result [2].

Cayley's Theorem: Every finite group G is isomorphic to a subgroup of the symmetric group $S_{|G|}$.

3.4. Summary and geometric intuition

The left regular action provides a powerful "external perspective": no matter how abstracts a group may be, it can always be realized as a concrete set of permutations acting on its own elements [8]. From the standpoint of the Orbit-Stabilizer Theorem, this action is structurally simple (a single orbit, trivial stabilizers), yet it proves the "representability" of the group. To visualize this, imagine the dihedral group D_3 of a triangle. Scatter its six elements as a set; the left regular action turns each group element into a "permutation table". Although these permutations are densely arranged, they form a subgroup of S_6. This is the first application of the Orbit-Stabilizer Theorem to a "macroscopic embedding" [9, 10].

4. The conjugation action and the class equation

Unlike the left regular action, which treated the group as a single monolithic block, the conjugation action fragments the group into internal layers defined by commutativity. This shift in perspective transforms the Orbit-Stabilizer Theorem from an embedding tool into a structural scalpel [7, 10].

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4.1. Construction of the conjugation action

For finite L, set $F = L$ again. The conjugation action (defined by inner automorphisms) is given by the rule

$$L \times L \to L,\ \ (\mathrm{m}, q) \mapsto m \cdot q = mqm^{-1}. \tag{23}$$

Then, verify the group action axioms:
Identity: $e \cdot q = eqe^{-1} = q$.
Compatibility: For $m_1, m_2 \in L$,

$$(m_1 m_2) \cdot q = (m_1 m_2) q (m_1 m_2)^{-1} = m_1 \left(m_2 q m_2^{-1}\right) m_1^{-1} = m_1 \cdot (m_2 \cdot q). \tag{24}$$

Thus, this is a valid action. However, its behavior is radically different from the left regular action. Here, each group element m induces the inner automorphism $\mathrm{Inn}_m : q \mapsto mqm^{-1}$, which preserves the algebraic structure of L rather than merely permuting its elements as a bare set [1, 10].

4.2. Orbits and stabilizers under conjugation

When applying the Orbit-Stabilizer Theorem to this action, the objects it produces have special names that lie at the heart of finite group theory.

4.2.1. Orbits as conjugacy classes

For an element $q \in L$, its orbit under conjugation is

$$O_q = \left\{hqh^{-1} \mid h \in L\right\}. \tag{25}$$

This subset is referred to as the conjugacy class of q, customarily written as $\mathrm{Cl}(q)$. Equivalently, $y \in \mathrm{Cl}(q)$ if and only if y is conjugate to q, i.e. $y = hqh^{-1}$ for some $h \in L$. The conjugacy classes are pairwise disjoint and their union is all of L [8, 10].

4.2.2. Stabilizers as centralizers

For an element $q \in L$, its stabiliser under conjugation is

$$\mathrm{Stab}(q) = \left\{h \in L \mid hqh^{-1} = q\right\}. \tag{26}$$

The condition $hqh^{-1} = q$ is equivalent to $hq = qh$, meaning that h commutes with q. Consequently, this stabilizer coincides with the centralizer of q in L, which is written as

$$C_L(q) = \{h \in L \mid hq = qh\}. \tag{27}$$

This centralizer forms a subgroup of L, and in fact contains the cyclic subgroup $\langle q \rangle$ [9, 10].

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4.2.3. Applying the orbit-stabilizer theorem

Applying the Orbit-Stabilizer Theorem to the conjugation action yields the fundamental identity:

$$|\mathrm{Cl}(q)| \cdot |C_L(q)| = |L| \tag{28}$$

or equivalently,

$$|\mathrm{Cl}(q)| = [L : C_L(q)]. \tag{29}$$

Thus, the size of a conjugacy class is precisely the index of the centralizer of any element in that class. This formula shows that large conjugacy classes correspond to elements with relatively small centralizers (i.e., elements that commute with few others), while elements in the centre of L have the smallest possible conjugacy class [1, 12].

4.3. The centre and singleton classes

A special case occurs when $\mathrm{Cl}(q) = \{q\}$. This happens exactly when $hqh^{-1} = q$ for all $h \in L$. Equivalently, when q commutes with every element of L. The elements having this property make up the group centre:

$$Z(L) = \{z \in L \mid zh = hz \text{ for all } h \in L\}. \tag{30}$$

Therefore, class size 1 iff $q \in Z(L)$. All non-central elements yield classes of size at least 2 [7, 8].

4.4. Derivation of the class equation

The partition of L into conjugacy classes allows to write L as a disjoint union of its centre and its non-singleton classes:

$$L = z(L) \cup \bigcup_i CI(q_i) \tag{31}$$

With q_i ranging over representatives of classes having size at least 2. Passing to cardinalities and applying $|\mathrm{Cl}(q)| = [L : C_L(q)]$ from (29) yields the Class Equation:

$$|L| = |Z(L)| + \sum_i [L : C_L(q_i)] \tag{32}$$

This formula is a cornerstone of finite group enumeration: the group's order decomposes into the centre's size plus the indices of proper centralizer subgroups [10-12].

4.5. An application: the non-trivial centre of finite p-group

The true strength of the Class Equation lies in its ability to impose strong restrictions on the structure of finite groups. In this part, applying it to a particularly important family of groups — the finite p -groups — to prove a foundational theorem that underpins much of the classification of finite groups [3, 11].

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4.5.1. Definition of a p-group

Given a prime p, a finite group L is a p -group when $|L|=p^n$, $n \geq 1$. Thus, every non-trivial subgroup has order divisible by p. This property imposes strong restrictions on the group structure, yet these groups appear frequently as the fundamental constituents of finite group theory [1].

4.5.2. Reduction of the class equation modulo p-groups

Define a non-trivial finite p b-group L with $|L|=p^n$ ($n \geq 1$). Write the Class Equation:

$$|L|=|Z(L)|+\sum_i [L: C_L(q_i)] \tag{33}$$

with q_i ranging over the non-central classes.

For each such non-central representative q_i, its centraliser $C_L(q_i)$ is a proper subgroup of L. By Lagrange's Theorem, the index $[L: C_L(q_i)]>1$. Moreover, since $|L|$ is a power of p, every proper subgroup has order a power of p, so its index is actually a positive power of p, hence divisible by p [11]. That is

$$p \mid [L: C_L(q_i)], \text{ for every non-central } q_i. \tag{34}$$

Now reduce the Class Equation modulo p:

$$|L|=|Z(L)|+\sum_i [L: C_L(q_i)] \quad (\operatorname{mod} p). \tag{35}$$

Since $|L|=p^n \equiv 0 \pmod p$ (for $n \geq 1$), and each term $[L: C_L(q_i)]$ is divisible by p, the summation vanishes modulo p [6]. Therefore, get

$$0 \equiv |Z(L)| \pmod p. \tag{36}$$

4.5.3. Conclusion: the centre is non-trivial

The congruence $|Z(L)| \equiv 0 \pmod p$ forces p to divide the order of the centre. Since $Z(L)$ is a subgroup, its order is at least 1 . Indeed, the identity element e always belongs to $Z(L)$, so $|Z(L)| \geq 1$. Under the divisibility condition p divides $|Z(L)|$, it must have $|Z(L)| \geq p$.

In particular, $Z(L)$ contains no fewer than p elements, and hence is strictly larger than the trivial subgroup $\{e\}$. Thus, the celebrated theorem has been proved [5, 13].

Theorem (Non-Trivial Centre of a p -Group): For any non-trivial finite p -group L, the centre $Z(L)$ is non-trivial. In fact, p divides $|Z(L)|$.

4.5.4. Significance and implications

This theorem is powerful though. It is particularly useful when classifying groups whose order are p^2, p^3, and beyond. As an illustration, consider the case $|L|=p^2$. The theorem forces $|Z(L)|$ to be either p or p^2. If it were p, then quotient $L/Z(L)$ would be cyclic, forcing L to be abelian (a standard lemma), a contradiction. Hence $Z(L)=L$. The conjugation-action version of the Orbit-Stabilizer Theorem is essential in this reasoning; without it, such a fundamental structural conclusion would be far more difficult to establish [10].

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5. Conclusion

This paper has explored the dual nature of the Orbit-Stabilizer Theorem by applying it to two fundamentally different actions of a group on itself. Under the left regular action, the theorem yields Cayley's embedding, providing an external representation of every abstract group as a concrete permutation group. Conversely, under the conjugation action, the theorem turns inward: orbits become conjugacy classes, stabilizers become centralizers, and the Orbit-Stabilizer formula gives rise to the Class Equation. This equation serves as a structural constraint, leading directly to the proof that every non-trivial finite p -group has a non-trivial centre. Thus, the same algebraic theorem, viewed through different lenses, simultaneously delivers an external realization and an internal dissection of a group, perfectly illustrating the interpretive flexibility and elegance of group theory. The present paper addresses only finite groups and does not treat character theory, or infinite cases, such extensions are left for the future.

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