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Counting Pentagon Colorings up to Symmetry: Burnside's Lemma and Its Weighted Form

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Abstract. This paper presents an application of the group actions and Burnside's lemma to the enumeration of vertex colorings of a regular pentagon. Each vertex receives one of three colors. Colorings related by any rotation or reflection of the pentagon are identified as equivalent. After reviewing the group actions, orbits, stabilizers, and the Orbit-Stabilizer theorem, Burnside's lemma is proved and applied to the action of the dihedral group D_5. The calculation shows that the $3^5 = 243$ possible colorings form the 39 equivalence classes under the symmetries of the pentagon. The weighted form of the Burnside's lemma is then introduced, and the invariant weight functions are used to refine this enumeration. In particular, the resulting generating polynomials classify the equivalence classes according to the number of vertices assigned a specified color and the number of monochromatic edges. These applications illustrate how weighted orbit counting provides more detailed combinatorial information than ordinary Burnside enumeration.

Keywords: Orbit-Stabilizer theorem, Burnside's lemma, weighted Burnside lemma, group actions, vertex colorings

1. Introduction

Consider coloring the vertices of a regular pentagon from a set of three colors. If the five vertices are treated as distinct and symmetries are ignored, the multiplication principle gives 3^5 possible colorings. The problem becomes less immediate, however, when colorings related by symmetries are regarded as equivalent. Counting configurations up to symmetry is a standard problem in combinatorial enumeration, for which group actions provide a useful method [1]. In the present example, two colorings are considered equivalent if one can be obtained from the other by rotations about the center of the pentagon or by reflections across one of its axes of symmetry. Thus, the counting problem is therefore reduced to finding the coloring orbits generated by the pentagon's symmetries.

Group actions provide a precise mathematical framework for describing such equivalence relations [2]. Let D_5 denote the symmetry group of the pentagon [3], acting on its vertex colorings through the corresponding permutations of the vertices. Two colorings are equivalent under rotations

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and reflections precisely when they lie in the same orbit of this action. Consequently, counting colorings up to symmetry is equivalent to determining the number of orbits under the action of D_5.

Two fundamental results support this approach. The Orbit–Stabilizer theorem expresses an orbit size through the corresponding stabilizer [4]. Burnside's lemma, or the orbit-counting lemma [5, 6] provides a direct method for counting orbits. The ordinary lemma, however, yields only the total number of orbits [7]. It does not show how many orbits have exactly k vertices of a specified color or exactly m monochromatic edges, where an edge is called monochromatic if its two endpoints are assigned the same color [8]. The weighted Burnside lemma addresses this limitation by assigning the same weight to equivalent colorings [9]. By choosing the weight to record either the number of vertices of a specified color or the number of monochromatic edges, one can classify the equivalence classes according to these two properties.

This paper gives an expository account of Burnside's Lemma and its weighted form through the example of three-colorings of the vertices of a regular pentagon. The discussion begins with the group-theoretic background needed for the counting argument. Burnside's lemma is then derived and used to separate the three-colorings into 39 orbits under the action of D_5. Finally, the paper states the weighted Burnside lemma and applies it using two invariant weight functions. The first records how many vertices are assigned a specified color, while the second records the number of monochromatic edges. The resulting weighted polynomials classify the same equivalence classes according to these two different properties.

2. Group actions and the Orbit-Stabilizer theorem

This section introduces some basic definitions and Lagrange's theorem, which are then used to prove the Orbit–Stabilizer theorem [10].

2.1. Basic definitions

Throughout this subsection, suppose that $\mathfrak{G}$ is a group and $\mathfrak{X}$ is a set.

Definition 2.1.1 (Group action). A left action assigns an element $g \bullet x$ of $\mathfrak{X}$ to every $g \in \mathfrak{G}$ and $x \in \mathfrak{X}$. This assignment must satisfy :

(a) $e \bullet x = x$;

(b) $(uv) \bullet x = u \bullet (v \bullet x)$,

for all $x \in \mathfrak{X}$ and $u,\ v \in \mathfrak{G}$, where e denotes the identity of $\mathfrak{G}$.

Definition 2.1.2 (Orbit and stabilizer). Suppose that $\mathfrak{G}$ acts on $\mathfrak{X}$ and choose $x \in \mathfrak{X}$. The orbit and stabilizer associated with x are defined, respectively, by $O_x = \{g \bullet x : g \in \mathfrak{G}\}$ and $\mathfrak{G}_x = \{g \in \mathfrak{G} :\ g \bullet x = x\}$.

2.2. The Orbit-Stabilizer theorem

Theorem 2.2.1 (Lagrange's theorem) [4]. Let $\mathfrak{G}$ be a finite group and let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$, written as $\mathfrak{H} \leq \mathfrak{G}$. Then the order of $\mathfrak{H}$ divides the order of $\mathfrak{G}$. Specifically:

$$|\mathfrak{G}| = [\mathfrak{G} : \mathfrak{H}] \cdot |\mathfrak{H}|, \tag{1}$$

where $[\mathfrak{G} : \mathfrak{H}]$ denotes the number of left cosets of $\mathfrak{H}$ in $\mathfrak{G}$.

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Proof. Since $\mathfrak{H} \leq \mathfrak{G}$, the left cosets of $\mathfrak{H}$ partition $\mathfrak{G}$. Then let $m = [\mathfrak{G} : \mathfrak{H}]$, and choose $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{G}$ as representatives of the distinct left cosets of $\mathfrak{H}$ in $\mathfrak{G}$. Then $\mathfrak{G}$ can be written as a disjoint union of m cosets:

$$G = \alpha_1 H \sqcup \alpha_2 H \sqcup \ldots \sqcup \alpha_m H, \tag{2}$$

where $\alpha_i \mathfrak{H} \cap \alpha_j \mathfrak{H} = \emptyset$ for $i \neq j$. Consider the following claim: every coset has the same size, namely $|\alpha_i \mathfrak{H}| = |e\mathfrak{H}| = |\mathfrak{H}|$.

To verify it, define a map $f : \mathfrak{H} \to \alpha_i \mathfrak{H}$ by $f(h) = \alpha_i h$. This map is surjective by the definition of cosets. It is injective since $\alpha_i h_1 = \alpha_i h_2$ implies $h_1 = h_2$ by left cancellation. Thus, $|\alpha_i \mathfrak{H}| = |\mathfrak{H}|$ for all i. Taking cardinalities in equation (2) gives:

$$|\mathfrak{G}| = \sum_{i=1}^{m} |\alpha_i \mathfrak{H}| = \sum_{i=1}^{m} |\mathfrak{H}| = m\,|\mathfrak{H}|. \tag{3}$$

Therefore, equation (3) yields $|\mathfrak{G}| = [\mathfrak{G} : \mathfrak{H}] \bullet |\mathfrak{H}|$. This is Lagrange's theorem, as expressed in equation (1).

Proposition 2.2.2. The stabilizer $\mathfrak{G}_x$ is a subgroup of $\mathfrak{G}$, namely:

$$\mathfrak{G}_x \leq G. \tag{4}$$

Proof. The subgroup criterion is verified as follows. The identity element belongs to $\mathfrak{G}_{\mathrm{x}}$ because $e \bullet x = x$. Suppose $g, u \in \mathfrak{G}_x$. Then $u^{-1} \cdot x = x$, and hence $(gu^{-1}) \cdot x = g \cdot x = x$. Therefore, $gu^{-1} \in \mathfrak{G}_x$. By the subgroup test, $\mathfrak{G}_x \leq \mathfrak{G}$.

If $\mathfrak{G}$ is finite, applying Lagrange's theorem as expressed in equation (1) to equation (4) yields

$$|\mathfrak{G}| = [\mathfrak{G} : \mathfrak{G}_x]|\mathfrak{G}_x|. \tag{5}$$

Theorem 2.2.3 (Orbit–Stabilizer theorem). Suppose a finite group $\mathfrak{G}$ acts on a set $\mathfrak{X}$. Then, for every $x \in \mathfrak{X}$,

$$|\mathfrak{G}| = |\mathfrak{G}_x||O_x|. \tag{6}$$

Equivalently, comparing equations (5) and (6) gives:

$$|O_x| = [\mathfrak{G} : \mathfrak{G}_x]. \tag{7}$$

Proof. Consider the map $f : \mathfrak{G}/\mathfrak{G}_x \to O_x, f(g\mathfrak{G}_x) := g \cdot x$ for $g \in \mathfrak{G}$, where $\mathfrak{G}/\mathfrak{G}_x$ denotes the set of left cosets of $\mathfrak{G}_x$ in $\mathfrak{G}$. It remains to show that this map is well-defined and bijective.

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1. Well-defined. It must be shown that if two cosets are equal, then their images are equal. Let u, $v \in \mathfrak{G}$ and suppose that $u\mathfrak{G}_x = v\mathfrak{G}_x$. Then $v^{-1}u \in \mathfrak{G}_x$, so $(v^{-1}u) \cdot x = x$. Therefore, $u \cdot x = v \cdot ((v^{-1}u) \cdot x) = v \cdot x$. Thus $f(u\mathfrak{G}_x) = f(v\mathfrak{G}_x)$, so f is well-defined.

2. Injective. Let u, $v \in \mathfrak{G}$ and suppose $f(u\mathfrak{G}_x) = f(v\mathfrak{G}_x)$. Then $u \cdot x = v \cdot x$, and therefore $(v^{-1}u) \cdot x = v^{-1} \cdot (u \cdot x) = v^{-1} \cdot (v \cdot x) = x$. Thus, $v^{-1}u \in \mathfrak{G}_x$, which implies $u\mathfrak{G}_x = v\mathfrak{G}_x$. Hence, f is injective.

3. Surjective. For any $y \in O_x$, the definition of O_x gives $y = g \cdot x$ for some $g \in \mathfrak{G}$. Hence, $y = f(g\mathfrak{G}_x)$, proving that f is surjective.

The map is therefore a bijection, so $|O_x| = [\mathfrak{G} : \mathfrak{G}_x]$, which proves equation (7). Combining equations (5) and (7) then gives equation (6).

3. Burnside's lemma and pentagon colorings

This section derives Burnside's lemma and uses the action of D_5 to enumerate the orbits of pentagon vertex colorings with three available colors.

3.1. Burnside's lemma

Theorem 3.1.1 (Burnside's lemma). Suppose that a finite group $\mathfrak{G}$ acts on a finite set $\mathfrak{X}$. Then for $|\mathfrak{X}/\mathfrak{G}| = |\{O_x | x \in \mathfrak{X}\}|$, the following equation can be obtained:

$$|X/G| = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} |\mathfrak{X}^g|, \tag{8}$$

where $\mathfrak{X}^g = \{x \in \mathfrak{X} | g \cdot x = x\}$ is the set of points fixed by g.

Proof. Consider the set of all fixed pairs $S = \{(g, x) \in \mathfrak{G} \times \mathfrak{X} | g \cdot x = x\}$. The number of elements in S is counted in two different ways.

1: Summing over $\mathfrak{G}$. For a fixed $g \in \mathfrak{G}$, the number of elements x such that $(g, x) \in S$ is exactly $|\mathfrak{X}^g|$. Therefore:

$$|S| = \sum_{g \in \mathfrak{G}} |\mathfrak{X}^g|. \tag{9}$$

2: Summing over $\mathfrak{X}$. For a fixed $x \in \mathfrak{X}$, the number of elements g such that $(g, x) \in S$ is exactly the size of the stabilizer $\mathfrak{G}_x$. Therefore:

$$|S| = \sum_{x \in \mathfrak{X}} |\mathfrak{G}_x|. \tag{10}$$

Equating equations (9) and (10) gives:

$$\sum_{g \in \mathfrak{G}} |\mathfrak{X}^g| = \sum_{x \in \mathfrak{X}} |\mathfrak{G}_x|. \tag{11}$$

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Let $\Omega_1, \Omega_2, \ldots, \Omega_k$ be the distinct orbits of $\mathfrak{X}$ (note that $k = |\mathfrak{X}/\mathfrak{G}|$). Then for any $x \in \mathfrak{X}$, $x \in \Omega_{i_x}$ for a unique $1 \le i_x \le k$. Therefore, comparing equations (11) and (6) gives:

$$\sum_{g\in\mathfrak{G}}|\mathfrak{X}^g| = \sum_{x\in\mathfrak{X}}|\mathfrak{G}_x| = \sum_{x\in\mathfrak{X}}\frac{|\mathfrak{G}|}{|O_x|} = |\mathfrak{G}|\sum_{i=1}^{k}\sum_{x\in\Omega_i}\frac{1}{|\Omega_i|}. \tag{12}$$

For any specific orbit Ω_i, the inner sum is just adding the value $\frac{1}{|\Omega_i|}$ to itself $|\Omega_i|$ times:

$$\sum_{x\in\Omega_i}\frac{1}{|\Omega_i|} = |\Omega_i|\cdot\frac{1}{|\Omega_i|} = 1. \tag{13}$$

Therefore, using equation (13), equation (12) reduces to

$$\sum_{g\in\mathfrak{G}}|\mathfrak{X}^g| = |\mathfrak{G}|\sum_{i=1}^{k}1 = |\mathfrak{G}|\cdot k = |\mathfrak{G}|\cdot|X/G|. \tag{14}$$

Hence,

$$|X/G| = \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}|\mathfrak{X}^g|. \tag{15}$$

3.2. Counting pentagon colorings

Let $\mathfrak{X}$ consist of all assignments of three available colors to the five pentagon vertices. The group $\mathfrak{G} = D_5$ has ten elements and acts on $\mathfrak{X}$ through the symmetries of the pentagon. Consequently, there are $|\mathfrak{X}| = 3\times3\times3\times3\times3 = 3^5 = 243$ elements in $\mathfrak{X}$. If one coloring can be obtained from another by rotations or reflections, then they are in the same orbit under this action. Thus, it remains to calculate $|\mathfrak{X}/\mathfrak{G}|$.

Burnside's lemma, as stated in equation (15), gives $|\mathfrak{X}/\mathfrak{G}| = \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}|\mathfrak{X}^g|$, so it remains to calculate $|\mathfrak{X}^g|$ for each $g \in D_5$. A coloring $x \in \mathfrak{X}$ is fixed by $g \in D_5$ if all vertices in the same cycle of the permutation g have the same color. If g has k cycles, there are 3^k fixed colorings. Therefore, the numbers of fixed colorings are summarized in Table 1.

Table 1. Fixed colorings under the action of D_5

Element Type	No. of Elements	No. of Cycles	$\lvert X^g\rvert$
Identity	1	5	3^5=243
Reflections	5	3	3^3=27
Nonidentity Rotations	4	1	3^1=3

Applying equation (8), the required orbit count is:

$$|X/G| = \frac{1}{10}(1\bullet243 + 5\bullet27 + 4\bullet3) = \frac{390}{10} = 39. \tag{16}$$

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Thus, the action of D_5 partitions the 243 colorings into 39 orbits.

4. Weighted Burnside's lemma and applications to pentagon colorings

The ordinary Burnside lemma determines only the total number of orbits. To classify the orbits according to additional combinatorial properties, this section introduces the weighted Burnside lemma and applies it to the same three-coloring problem.

4.1. Weighted Burnside's lemma

Lemma 4.1.1 (Weighted Burnside Lemma; Lemma 14.3 in [11]). Let a finite group $\mathfrak{G}$ act on a finite set $\mathfrak{X}$, and let $w:\ \mathfrak{X} \to R$ be a weight function, where R is a commutative ring in which $|\mathfrak{G}|$ is invertible. Assume that

$$w(g \cdot x) = w(x) \tag{17}$$

for every $g \in \mathfrak{G}$ and $x \in \mathfrak{X}$. A weight function satisfying equation (17) is called $\mathfrak{G}$ -invariant. Thus, w is constant on each orbit; for an orbit O, denote its common value by $w(O)$. For $g \in \mathfrak{G}$, write

$$\mathfrak{X}^g \ = \ \{x \in \mathfrak{X}:\ g \cdot x = x\}. \tag{18}$$

With $\mathfrak{X}^g$ as defined in equation (18), the orbit-weight sum is

$$\sum_{O \in \mathfrak{X}/\mathfrak{G}} w(O) \ = \ \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \sum_{x \in \mathfrak{X}^g} w(x). \tag{19}$$

If $w(x) = 1$ for every $x \in \mathfrak{X}$, then equation (19) reduces to the ordinary Burnside formula in equation (8). Therefore, the ordinary Burnside lemma is a special case of the weighted Burnside lemma.

For the present application, $\mathfrak{X}$ is the set of all three-colorings of the vertices of a regular pentagon, and $\mathfrak{G} = D_5$ acts on $\mathfrak{X}$ by rotations and reflections. Two $\mathfrak{G}$ -invariant weight functions will be considered. The first records the number of vertices assigned a specified color, while the second records the number of monochromatic edges. For the polynomial applications below, take $R = \mathbb{Q}[t]$, where $\mathbb{Q}[t]$ denotes the polynomial ring over the rational numbers and t is an indeterminate.

4.2. Classification by the number of vertices of a specified color

Fix one of the three colors, denoted by A. For a coloring $\varphi \in \mathfrak{X}$, let $v(\varphi)$ be the number of vertices assigned color A. Define the weight function

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$$w_1(\varphi) = t^{v(\varphi)}. \tag{20}$$

The value $v(\varphi)$ is unchanged by rotations and reflections. Therefore, the weight function in equation (20) is D_5 -invariant, and the Weighted Burnside Lemma (19) can be applied. Since v is constant on each orbit, write $v(O) = v(\varphi)$ for any $\varphi \in O$.

For the identity element e, each of the five vertices may be assigned color A, contributing weight t, or one of the other two colors, contributing total weight 2. Hence, the fixed-coloring weight sum is

$$\sum_{\varphi \in \mathfrak{X}^e} t^{v(\varphi)} = (t+2)^5. \tag{21}$$

Every nonidentity rotation r consists of a single cycle of length five. A coloring fixed by r must assign the same color to all five vertices. Therefore,

$$\sum_{\varphi \in \mathfrak{X}^r} t^{v(\varphi)} = t^5 + 2. \tag{22}$$

Every reflection s has one fixed vertex and two cycles of length two. The fixed vertex contributes $t + 2$, while each two-cycle contributes t^2+2. Thus,

$$\sum_{\varphi \in \mathfrak{X}^s} t^{v(\varphi)} = (t+2)(t^2+2)^2. \tag{23}$$

Since D_5 contains one identity element, four nonidentity rotations, and five reflections, substituting equations (21) – (23) into the Weighted Burnside Lemma (19) gives

$$\sum_{O \in \mathfrak{X}/D_5} t^{v(O)} = \tfrac{1}{10}\left[(t+2)^5 + 4(t^5+2) + 5(t+2)(t^2+2)^2\right]. \tag{24}$$

Expanding equation (24) gives

$$\sum_{O \in \mathfrak{X}/D_5} t^{v(O)} = 8 + 10t + 12t^2 + 6t^3 + 2t^4 + t^5. \tag{25}$$

In equation (25), the coefficient attached to t^k counts the orbits with exactly k such vertices. In particular, evaluating equation (25) at $t = 1$ gives 39, agreeing with the result obtained from the ordinary Burnside calculation in equation (16).

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4.3. Classification by the number of monochromatic edges

A second application classifies the equivalence classes according to the number of monochromatic edges. For a coloring $\varphi \in \mathfrak{X}$, let $m(\varphi)$ denote the number of edges whose endpoints receive the same color. Define the weight function by

$$w_2(\varphi) = t^{m(\varphi)}. \tag{26}$$

Every element of D_5 preserves adjacency and the colors assigned to the vertices, so $m(g \cdot \varphi) = m(\varphi)$ for every $g \in D_5$. Hence, the weight function in equation (26) is D_5 -invariant, and the Weighted Burnside Lemma (19) can be applied. Since m is constant on each orbit, write $m(O) = m(\varphi)$ for any $\varphi \in O$.

For each $g \in D_5$, define the fixed-coloring polynomial

$$F_g(t) = \sum_{\varphi \in \mathfrak{X}^g} t^{m(\varphi)}. \tag{27}$$

Thus, the subscript in $F_g(t)$ identifies the group element whose fixed colorings are being counted.

The identity element e fixes all 3^5 colorings. To obtain the distribution, let $q = 5 - m(\varphi)$ be the number of color changes around the cycle. After choosing the q change-edges, the resulting cyclic color-runs must receive adjacent distinct colors. For $q = 5,\ 4,\ 3$, and 2, this gives 30, $5 \times 18 = 90$, $10 \times 6 = 60$, and $10 \times 6 = 60$ colorings, respectively; $q = 1$ is impossible, while $q = 0$ gives the three constant colorings. Therefore, definition (27) gives

$$F_e(t) = 30 + 90t + 60t^2 + 60t^3 + 3t^5. \tag{28}$$

Every nonidentity rotation r fixes only the three constant colorings. In each of these colorings, all five edges are monochromatic. Hence, definition (27) gives

$$F_r(t) = 3t^5. \tag{29}$$

Now let s be a reflection. It fixes one vertex and interchanges the remaining four vertices in two pairs. A coloring fixed by s is determined by three colors a, b, and c, arranged around the pentagon as $(a,\ b,\ c,\ c,\ b)$. The edge joining the two vertices colored c is always monochromatic. If $a = b$, two additional edges are monochromatic, and if $b = c$, another two edges are monochromatic.

There are 3 fixed colorings with $a = b = c$, giving five monochromatic edges; 12 with exactly one of $a = b$ and $b = c$, giving three; and 12 with $a \neq b$ and $b \neq c$, giving one. Taking $g = s$ in definition (27) explains the notation $F_s(t)$ and gives

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$$F_s(t) = 12t + 12t^3 + 3t^5. \tag{30}$$

Since D_5 contains the identity, four nonidentity rotations, and five reflections, the Weighted Burnside Lemma (19) and definition (27) give

$$\sum_{O \in \mathfrak{X}/D_5} t^{m(O)} = \frac{1}{10}\left[F_e(t) + 4F_r(t) + 5F_s(t)\right]. \tag{31}$$

Substituting equations (28) – (30) into equation (31) and simplifying gives

$$\sum_{O \in \mathfrak{X}/D_5} t^{m(O)} = 3 + 15t + 6t^2 + 12t^3 + 3t^5. \tag{32}$$

Equation (32) shows that there are 3 equivalence classes with no monochromatic edges, 15 with one, 6 with two, 12 with three, none with four, and 3 with five. Evaluating equation (32) at $t = 1$ gives 39, which agrees with the total number of equivalence classes obtained from the ordinary Burnside calculation in equation (16).

5. Conclusion

This paper has examined the enumeration of three-colorings of the vertices of a regular pentagon under rotations and reflections. By interpreting the symmetries of the pentagon as an action of the dihedral group D_5, the coloring problem was reformulated as a problem of counting orbits. The Orbit–Stabilizer Theorem provided the necessary connection between orbits and stabilizers, while Burnside's lemma converted the orbit-counting problem into a calculation involving the colorings fixed by each symmetry. This calculation showed that the $3^5 = 243$ vertex colorings form 39 equivalence classes.

The weighted Burnside lemma provided a refinement of this result. When the weight records the number of vertices assigned a specified color, the orbit-generating polynomial is $8 + 10t + 12t^2 + 6t^3 + 2t^4 + t^5$. When the weight records the number of monochromatic edges, the corresponding polynomial is $3 + 15t + 6t^2 + 12t^3 + 3t^5$. In each case, the sum of the coefficients is 39, which agrees with the result obtained from the ordinary Burnside lemma. However, the coefficients provide additional information by describing how the equivalence classes are distributed according to the chosen statistic.

Thus, ordinary Burnside enumeration determines the total number of equivalence classes, whereas its weighted form reveals more detailed structural information about those classes. The pentagon-coloring example therefore illustrates both the effectiveness of group actions in counting objects up to symmetry and the usefulness of weighted orbit counting in refined combinatorial enumeration.

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