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Frobenius Coin Problem from Two Coins Denominations to K Coins

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Abstract. The Frobenius Coin Problem is a classic mathematical puzzle that asks for the largest unrepresentable integer using a given set of relatively prime coin values. Beyond finding this maximum number, exploring how the number of combination solutions grows is equally important. This paper focuses on calculating the asymptotic growth rate of these combinations, moving from the basic two-coin case to a general k-coin scenario. Instead of relying on pure algebraic estimations, which easily fail for three or more coins, this study uses a continuous complex analysis framework. The discrete counting problem is converted into generating functions, and Cauchy's Residue Theorem is applied. By analyzing the dominant highest-order pole at z=1 and using the Leibniz rule, the asymptotic growth rate for k coins is strictly determined as $n^{k-1}/(k-1)!a_1 \ldots a_k$. This analytical approach successfully overcomes the dimensional limitations of traditional algebra. Ultimately, it provides a systematic blueprint for understanding combination behaviors in broader linear Diophantine equations.

Keywords: Frobenius Coin Problem, Complex Analysis, Generating Function

1. Introduction

The Frobenius Coin Problem (also called the linear Diophantine problem of Frobenius) is a very popular subject of study in number theory. It is about the largest integer that is not a linear combination of coin values with some of which are coprime. In 1884, James J. Sylvester discovered the first big step forward by determining the exact maximum unrepresentable integer in the case of two coins [1]. But this issue is not limited to just coin changes. It has evolved over the years to become a basic tool of commutative algebra and discrete mathematics [2]. In addition, knowledge of the limits of such linear combinations is useful in real-life applications of computer science, such as in public-key cryptography and algorithmic design [3].

The general k-coin problem has been the subject of intensive research by mathematicians in recent years. The exact Frobenius number problem for large sets is mathematically known to be an NP-hard problem [4]. Since there does not exist a universal algebraic formula in the case of three or more coins, recent literature often concentrates on alternative approaches. For example, contemporary researchers have studied the algebraic properties of numerical semigroups to prove some sharp bounds for the integers that are not in the semigroup [5]. Some other authors have investigated exact formulas of particular integer sequences [6, 7]. Moreover, the average behavior of

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the Frobenius number has been the subject of an increasing number of studies, based on probabilistic models [8]. In particular, recent works are devoted to the exact solutions of the Frobenius coin problem in some finite cases [9]. In the modern trends, one of the most effective is that the use of continuous mathematical tools, to analyze the asymptotic behavior of combination solutions [10]. This particular mode of analysis is first and foremost the inspiration for the research.

This paper will try to demonstrate the application of generating functions and complex analysis in solving Frobenius Coin Problem. The organization of this paper is as below. Section 2 presents the two-coin case, and sets up an algebraic check and then uses complex analysis to verify. In section 3, this framework is extended to an n number of k coins. The poliness of the generating functions is used to derive a universal formula for the combination growth rate by using the Leibniz rule. Finally, a short conclusion of the analysis results is given in Section 4.

2. Method and Theory

2.1. Mathematical preliminaries

To begin with, this paper shall introduce the generating function. It is a formal power series used to encode a sequence of numbers. If there is a sequence c_n, its generating function is $G(x) = \sum_{n=0}^{\infty} c_n x^n$. In combinatorics, when people combine two independent choices to form a sequence, the product rule is applied by treating the combined function as a product of two independent generating functions.

In addition, this paper needs to use some tools from complex analysis, such as Poles and Order. In complex analysis, if a function $f(z)$ goes to infinity at $z = z_0$, mathematicians say z_0 is a pole. If n is the least positive integer such that $\lim_{z \to z_0} (z - z_0)^n f(z)$ is finite, then the pole is of order n. Another important concept is the Residue. When a complex function is expanded into a Laurent series around a pole z_0, the coefficient of the $(z - z_0)^{-1}$ term is called the residue. It represents the core value used in contour integration. This leads to Cauchy's Residue Theorem, which is a theorem stating that the contour integral of a function along a closed path is equal to $2\pi i$ times the sum of the residues of the poles inside the path.

Finally, the author will discuss Asymptotic Growth, using the notation $f(n) \sim g(n)$ to describe the behavior of a function as n goes to infinity. It means the limit of $f(n)/g(n)$ approaches 1 as $n \to \infty$.

2.2. Algebraic proof for the largest unrepresentable integer

People can solve the maximum unrepresentable integer for two coins with pure algebra. This paper wants to prove that the largest integer that cannot be written as $ax + by$ (for $x, y \geq 0$) is exactly $ab - a - b$. Assume that $ab - a - b$ can be represented. That means there exist non-negative integers x and y such that: $ab - a - b = ax + by$ and $ab = a(x + 1) + b(y + 1)$. Since a and b are coprime, a must divide $b(y + 1)$. Because $\gcd(a, b) = 1$, a must divide $y + 1$. Similarly, b must divide $x + 1$. Since $x \geq 0$ and $y \geq 0$, one knows that $x + 1 \geq 1$ and $y + 1 \geq 1$. For a to divide $y + 1$, the smallest possible value for $y + 1$ is a. For b to divide $x + 1$, the smallest possible value for $x + 1$ is b. Therefore,

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$$a(x+1)+b(y+1)\geq ab+ba=2ab \tag{1}$$

However, this contradicts the assumption $ab=a(x+1)+b(y+1)$, because $2ab$ is strictly greater than ab. Therefore, the initial assumption must be false. The number $ab-a-b$ cannot be represented by $ax+by$. Furthermore, it is a known algebraic fact that every integer strictly greater than $ab-a-b$ can be represented.

Now the author will prove that for any integer $n > ab-a-b$, there exist non-negative integers x and y such that $ax+by=n$. Because $\gcd(a,b)=1$, Bézout's identity states that there always exist integers x' and y' (which can be negative) such that $ax^{'}+by^{'}=n$. All possible solutions to this equation can be written in the form: $x=x^{'}+bk$ and $y=y^{'}-ak$ for any integer k.

One can always find a k that makes x fall between 0 and $b-1$ inclusive. This specific solution is named x_0 and y_0. Thus, $0\leq x_0\leq b-1$ and $n=ax_0+by_0$ Now, it is necessary to show that for this specific solution, y_0 must also be non-negative. The equation can be rewritten as $by_0=n-ax_0$. Given that $n > ab-a-b$, which means $n\geq ab-a-b+1$, and knowing the maximum possible value for x_0 is $b-1$, one can find the minimum value for by_0 : $by_0\geq(ab-a-b+1)-a(b-1)$, $by_0\geq ab-a-b+1-ab+a$, $so\, by_0\geq -b+1$ Since y_0 is an integer, the inequality $by_0\succ b$ means $y_0\succ 1$. Therefore, it must be true that $y_0\geq 0$. This proves that for any $n > ab-a-b$, one can successfully find non-negative integers x_0 and y_0. Thus, $ab-a-b$ is the strictly largest integer that cannot be formed by coins a and b.

2.3. Converting to generating function

While the algebraic proof finds the Frobenius number for two coins, it does not tell the number of ways to represent a given integer n, which is denoted as $r(n)$. To calculate $r(n)$, the author converts the problem into a generating function $R(x)$. Following the product rule, overall generating function can be treated strictly as a product of two independent generating functions for the two coins. The generating function for choosing multiples of a is $\sum x^{ak}=\frac{1}{1-x^a}$, and for b is $\sum x^{bj}=\frac{1}{1-x^b}$. Multiplying them gives:

$$R(x)=\sum_{n=0}^{\infty}r(n)x^n=\frac{1}{(1-x^a)(1-x^b)} \tag{2}$$

Before diving into the complex analysis method to find the asymptotic, one can use an algebraic method to estimate the growth rate of $r(n)$, which is the number of solutions to $ax+by=n$ for non-negative integers x and y. The equation can be rewritten to solve for y: $y=\frac{n-ax}{b}$ Since y must be a non-negative integer, there must be two conditions satisfied. First, $n-ax$ must be greater than or equal to 0, which means x must satisfy $0\leq x\leq n/a$. This shows a finite range for the possible values of x.

Second, for y to be an integer, $n-ax$ must be perfectly divisible by b, which means $ax\equiv n(modb)$. Because a and b are relatively prime, a has a unique multiplicative inverse modulo b. This means that for every b consecutive integers, there is exactly one integer x that satisfies the equation $ax\equiv n(modb)$. Now, one can estimate the number of valid solutions. The

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total length of the range for x is n/a. Since exactly one valid x appears in every block of size b, the author can find the total number of solutions by dividing the range by b:

$$r(n) \approx \frac{n/a}{b} = \frac{n}{ab} \tag{3}$$

Hence, using the algebraic technique, one gets that the asymptotic growth rate of the combinations is:

$$r(n) \sim \frac{n}{ab} \tag{4}$$

The algebraic estimation works for two coins. This approach is very complicated, however, if there are three or more coins involved.

2.4. Complex analysis calculation for the asymptotic

By definition, $r(n)$ is the coefficient of the z^n term in this series. According to Cauchy's integral formula, the author extracts this specific coefficient by calculating the residue of the function $\frac{R(z)}{z^{n+1}}$ at the pole $z = 0$. Therefore, there is $r(n) = \text{Res}_{z=0}$. Instead of calculating $\text{Res}_{z=0}$ directly, this paper uses a powerful technique from complex analysis. Suppose the function is integrated along a massive circular contour $|z| = R$ that encloses all the poles in the complex plane. According to Cauchy's Residue Theorem, this integral equal $2\pi i$ times the sum of all residues inside the circle. The poles of the function are located at $z = 0$, and the roots of unity from $1 - z^a = 0$ and $1 - z^b = 0$.

To evaluate this contour integral as the radius R approaches infinity, it is helpful to use the ML inequality. This theorem states that the absolute value of an integral is bounded by the maximum value of the function on the contour (M) multiplied by the length of the contour (L). For the circular path$|z| = R$, the length is simply the circumference, $L = 2\pi R$. To find the maximum value M, one can apply the triangle inequality to bound the denominator. Since $|1 - z^a| \geq |z|^a - 1 = R^a - 1$ and $|1 - z^b| \geq R^b - 1$, the absolute value of the complex function is strictly bounded by $M = \frac{1}{(R^a-1)(R^b-1)R^{n+1}}$.

By multiplying M and L, the absolute value of the entire contour integral is less than or equal to: $M \times L = \frac{2\pi R}{(R^a-1)(R^b-1)R^{n+1}}$. As R approaches infinity, the denominator grows at a rate of $R^{a+b+n+1}$, which is significantly faster than the linear growth of the numerator R. This forces the entire fraction, and therefore the integral itself, to approach exactly 0. Because the value of this contour integral is 0, the sum of all enclosed residues must also be exactly 0. This brings the equation: $\text{Res}_{z=0} + \text{Res}_{z=1} + \sum_{z_k \neq 1} \text{Res}_{z=z_k} = 0$ Because $r(n) = \text{Res}_{z=0}$, now there is

$$r(n) = -\text{Res}_{z=1} - \sum \text{Res}_{\text{other roots}} \tag{5}$$

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To find the poles, the author sets the denominator to zero, which gives the a-th and b-th roots of unity. Because a and b are relatively prime, the only common root is $z = 1$. This makes $z = 1$ a pole of order 2, while all other roots of unity are simple poles of order 1. As n approaches infinity, the dominant contribution to the formula comes exclusively from this highest-order pole at $z = 1$. Here is how to calculate the residue at $z = 1$ using the standard limit formula for a second-order pole

$$\mathrm{Res}_{z=1}\left(\frac{1}{(1-z^a)(1-z^b)z^{n+1}}\right) = \lim_{z\to 1}\left(\frac{d}{dz}\left(\frac{(z-1)^2}{(1-z^a)(1-z^b)z^{n+1}}\right)\right) \tag{6}$$

To simplify the calculation, the author introduces two finite geometric series polynomials, $A(z)$ and $B(z)$. They are defined as follows: $A(z) = \frac{1-z^a}{1-z} = 1 + z + z^2 + \ldots + z^{a-1}$ and $B(z) = \frac{1-z^b}{1-z} = 1 + z + z^2 + \ldots + z^{b-1}$ By substituting $A(z)$ and $B(z)$ into the limit, the $(z-1)^2$ term in the numerator cancels out the $(1-z)^2$ from the denominators of $A(z)$ and $B(z)$. The residue limit formula becomes

$$\lim_{z\to 1}\left(\frac{d}{dz}\left(\frac{1}{A(z)B(z)z^{n+1}}\right)\right) = \lim_{z\to 1}\left(\frac{d}{dz}\left(A(z)^{-1}B(z)^{-1}z^{-(n+1)}\right)\right) \tag{7}$$

Now the product rule for differentiation can be used. When people take the derivative of these three multiplied together terms, one gets three separate terms:

$$\lim_{z\to 1}\left[-\frac{n+1}{A(z)B(z)z^{n+2}} - \frac{A'(z)}{A(z)^2B(z)z^{n+1}} - \frac{B'(z)}{A(z)B(z)^2z^{n+1}}\right] \tag{8}$$

To evaluate this limit at $z = 1$, one needs to find the values of $A(1)$, $B(1)$, $A'(1)$, and $B'(1)$. From previous definitions, plugging in $z = 1$: $A(1) = 1 + 1 + \ldots + 1 = a$, $B(1) = 1 + 1 + \ldots + 1 = b$, $A'(1) = 1 + 2 + 3 + \ldots + (a-1) = \frac{a(a-1)}{2}$ and $B'(1) = 1 + 2 + 3 + \ldots + (b-1) = \frac{b(b-1)}{2}$. Substituting all these specific values back into the differentiated equation, there is: $-\frac{n+1}{ab} - \frac{\frac{a(a-1)}{2}}{a^2b} - \frac{\frac{b(b-1)}{2}}{ab^2}$, which equals

$$-\frac{n}{ab} - \frac{a+b}{2ab} \tag{9}$$

This is the exact residue at the dominant pole $z = 1$. For very large values of n, the constant term $\frac{a+b}{2ab}$ becomes negligible compared to the term containing n. Therefore, the complex analysis calculation strictly concludes that the asymptotic growth rate of the combinations is:

$$r(n) \sim \frac{n}{ab} \tag{10}$$

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This is to check the answer found algebraically. Furthermore, it also serves as a basis for learning Frobenius Coins Problems with 3 or more coins, which the pure algebraic method does not have.

3. Generalizing to more than two types of coin

3.1. Problem of restriction to 3 or more coins with the algebraic approach

Before applying complex analysis to multiple coins, it is necessary to understand why the previous algebraic estimation fails for three or more denominations. For the two-coin case, the algebraic method successfully estimates the growth rate by solving $ax + by = n$ for y, yielding $y = \frac{n-ax}{b}$ This creates a simple linear range $0 \leq x \leq n/a$, and a single modular condition a ≡ n (mod b). Because a and b are coprime, exactly one valid integer x exists in every block of size b, making the total number of solutions simply the length of the range divided by b. However, introducing a third coin c breaks this simple estimation. For the equation $ax + by + cz = n$, solving for z gives $z = \frac{n-ax-by}{c}$.

For z to be a non-negative integer, two conditions must be met. First, $ax + by \leq n$, which forms a two-dimensional triangular region rather than a simple linear interval. Second, the modular condition becomes much more complicated: $ax + by \equiv n(modc)$. Counting the exact number of valid (x, y) pairs that satisfy this specific congruence within a triangular boundary is mathematically extremely difficult. The logic of simply dividing an interval by a constant no longer applies to a two-dimensional area. As the number of coins increases to k, this becomes a $(k-1)$ -dimensional volume problem. Therefore, the algebraic method cannot be generalized, and complex analysis must be used.

3.2. General k-coin generating function

To find a universal formula for k coins ($a_1, a_2, \ldots, a_k$) where $\gcd(a_1, \ldots, a_k) = 1$, the problem is converted into a generating function. Similar to what was done for the two-coin case, the overall generating function R(z) is the product of the individual infinite series for each value of coin:

$$R(z) = \sum_{n=0}^{\infty} r_k(n) z^n = \prod_{i=1}^{k} \frac{1}{1-z^{a_i}} \tag{11}$$

To isolate the coefficient $r_k(n)$, Cauchy's integral formula is applied to $\frac{R(z)}{z^{n+1}}$ around a circular contour with very large radius. Exactly like the two-coin case, as the radius of this contour approaches infinity, the degree of the denominator exceeds the numerator, causing the entire integral to approach zero. Therefore, the sum of all residues must equal zero, which means $r_k(n)$ is determined by the negative sum of the residues at all other poles. The poles are found at the roots of unity for each $1 - z^{a_i} = 0$. Because the k integers are relatively prime, the only root shared by all denominators is z=1. Therefore, z=1 is a pole of order k. All other roots of unity have an order less than k, meaning that they are lower-order poles. As n approaches infinity, the dominant contribution to the combination growth rate is exclusively from the highest-order pole at z=1.

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3.3. Asymptotic growth rate for k coins

The negative residue at $z = 1$ has to be calculated, in order to determine the asymptotic growth rate. It is a pole of order k and you have to take the derivative (k-1) times:

$$r_k(n) \sim -\frac{1}{(k-1)!} \lim_{z\to 1}\left[\frac{d^{k-1}}{dz^{k-1}}\left(\frac{(z-1)^k}{(1-z^{a_1})\ldots(1-z^{a_k})z^{n+1}}\right)\right] \tag{12}$$

As in the two-coin case, a finite polynomial A(z) is added to cancel out the zero-factors in order to process this huge derivative

$$A(z) = \prod_{i=1}^{k} \frac{1-z}{1-z^{a_i}} \tag{13}$$

This reduces to:

$$-\frac{1}{(k-1)!} \lim_{z\to 1}\left[\frac{d^{k-1}}{dz^{k-1}}\left(A(z)^{-1} z^{-n-1}\right)\right] \tag{14}$$

Applying the Leibniz rule for repeated differentiation of a product allows one to systematically calculate this derivative. The most significant contribution appears when the term z^{-n-1} is differentiated the maximum number of times, which is $k-1$ times. Each time a derivative is taken with respect to z, a factor containing n is multiplied down, creating a polynomial of degree k-1 in terms of n. Evaluating this dominant derivative term at z=1 produces a binomial coefficient structure:

$$\mathrm{Res}_{z=1} = -\frac{1}{a_1\ldots a_k}\binom{n+k-1}{k-1} + O(n^{k-2}) \tag{15}$$

The remaining lower-order terms and the residues from other roots of unity contribute at most $O(n^{k-2})$, which can be ignored for very large values of n. Finally, by expanding the binomial coefficient $\binom{n+k-1}{k-1}$, the highest power of n is isolated as $\frac{n^{k-1}}{(k-1)!}$. Since $r_k(n)$ equals the negative of this residue, the negative signs cancel out. The final asymptotic growth rate for an arbitrary number of k coins is strictly determined as:

$$r_k(n) \sim \frac{n^{k-1}}{(k-1)! a_1 a_2 \ldots a_k}. \tag{16}$$

4. Conclusion

The Frobenius Coin Problem is not just a classical math puzzle. Instead, it provides a great opportunity to connect discrete combinatorics with continuous complex analysis. This paper mainly

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focused on finding the asymptotic growth rate of the combination solutions. By starting with the basic two-coin case, an algebraic baseline was first established. However, since algebraic estimations become extremely difficult for three or more coins, the analysis shifted to a complex analysis method. Through generating functions and Cauchy's Residue Theorem, the discrete counting problem was transformed into a limit calculation. Specifically, by focusing on the highest-order pole at z=1 and applying the Leibniz rule, it was clearly shown that the combination growth rate for k coins asymptotically approaches $\frac{n^{k-1}}{(k-1)!a_1 \ldots a_k}$.

For future research, there are still several interesting directions to explore. While the dominant growth term is now clear, further studies could look closer at the lower-order terms, such as the $O(n^{k-2})$ elements, to build a more specific mathematical formula. Moreover, future studies might use advanced algorithms or multi-dimensional geometric models to find strict upper bounds for those maximum unrepresentable integers of three or more coins. Ultimately, the complex analysis techniques discussed here can serve as a highly useful tool for solving broader linear Diophantine equations.

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