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Group Action, Orbit-Stabilizer Theorem and Burnside's Lemma

Ruichao Xia

Dulwich International High School Program Hengqin, Zhuhai, China
ruichao.xia@outlook.com

Abstract. This paper offers a thorough exploration of group action, equivalence class, and their utilization in combinatorial counting problems. Starting with the fundamentals of group theory and set relations, this section encompasses definitions and proofs of elementary results. Then, it proceeds to provide a proof of Lagrange's Theorem by illustrating that orbits divide the set into distinct equivalence classes through the use of an equivalence relation within the set. Following Lagrange's Theorem, this paper establishes the Orbit-Stabilizer Theorem by demonstrating that stabilizers partition the entire group and proving that each stabilizer contains the same number of elements. This paper subsequently demonstrates Burnside's Lemma, a tool employed for determining the quantity of orbits under a particular group action. This is achieved by utilizing the Orbit-Stabilizer Theorem and more profound insights derived from an equivalence relation concerning partitions. Therefore, this approach can be utilized to tackle the face coloring issue on a cube. Analyzing the various operations and determining the number of configurations they resolve. In conclusion, this study places these findings in a broader algebraic framework, emphasizing their ties to Pólya's Enumeration Theorem and computational group algorithms.

Keywords: Group action, relation, orbit, stabilizer

1. Introduction

Group theory provides a mathematical language to describe symmetry provides powerful framework tools to solve structural, geometrical, and algebraic problems in mathematics [1, 2]. This paper focuses on group action, which demonstrates how an abstract group acts on a given set [3]. In short, this essay is about how to apply group theory to a given object.

Two fundamental concepts derived from group action are orbit and stabilizer [1]. The orbit of an element is the points they are able to reach under a group action; elements in the group that fix a certain point in the given set are called stabilizers of that point. This paper will write proof of the Orbit-Stabilizer theorem; This paper is going to show all the proof processes from the basic knowledge of group and set, including relation in set and Lagrange's Theorem in group theory.

In detail, this paper will establish the definitions of group, group actions, and equivalence class at the beginning. Later on, prove the foundational result of group action, such as a group acting on a set is the same as a homomorphism from the group to the symmetric group of that set [3, 4]. After

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that, this study further proves the Orbit-Stabilizer theorem by using these results, then extend it to orbit counting by proving Burnside's Lemma. Last but not least, this paper will show a short example of applying the Lemma to combinatorial enumeration problem.

2. Basic of group theory

2.1. Definition of group

A group is a structure consisting of a set G, a binary operation · defined on G , and an element e belonging to G, satisfying certain properties. For every t, y, and u in the group G , it holds that (t multiplied by y) multiplied by u is equal to t multiplied by (y multiplied by u). (associativity)

For every element α in the group G , multiplying it by the identity element e yields α itself.

For every element v in the group G, there exists an element i in G such that the product of v and i is equal to the product of i and v, which in turn equals the identity element e. (inverse) [1, 5].

2.2. Definition of group action

A group G performs an operation · on a set χ, which can be represented as a function * :A function $G \times \chi \to \chi$ adheres to the identity and associativity axioms as outlined in references [2] and [3].

For any element u in the set χ, the operation e applied to u yields u itself. (A1)

For every element y, t in G and u in χ, the operation (y · t) * u is equal to y * (t * u). (A2)

2.2.1. Example of group action

(i) Trivial Action

For all τ in G and γ in M, the equation τ * x = x holds true. Each component secures each location.

Therefore, for each element γ in the set M, τ can be chose to be the identity element e, such that the product of τ and γ is equal to the product of e and γ, which is simply γ. (A1)

For α, τ in G and γ in M, (α · τ) * γ equals γ, with α · τ also being an element of G. Therefore, the expression (α · τ) * γ is equivalent to γ = α * γ = α * (τ * γ) (A2)

(ii) Left Regular Action

X = G, action by left multiplication: u * u' = u · u'.

$$u*u' = u \cdot u', u*u' = e \cdot u' = u' \quad \text{(A1)}$$

$$\text{for } y, j \in G,\ (y \cdot j) * u' = (y \cdot j) \cdot u' = y \cdot (j \cdot u') = y \cdot (j * u') = y * (j * u') \quad \text{(A2)}$$

(iii) Symmetric Group Action

Sym(M) acts on M by evaluation: f * m = f(m).

e * m = e(m) = m, e is an identity map (A1)

$$\text{for } y, k \in G,\ (y \cdot k) * m = (y \cdot k)(m) = y(k(m)) = y \cdot (k \cdot m) \quad \text{(A2)}$$

(iv) Symmetries of a Solid

The symmetry group exerts its influence on the various elements of a solid, such as points, vertices, edges, and faces.

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(v) Dihedral Group Action

D_{2n} acts on the vertices of a regular n-gon.

2.2.2. Proposition

An group action * of a group G on a given set M is homologous to a homomorphism $\mathfrak{Ɔ} : G \to \mathrm{Sym}(M)$ [2, 4], with an action * and $s \in G$, now take a consideration of the function

$$t_s : M \to M, \quad m \mapsto s*m$$

Proof. There is a similar function $t_{s^{-1}}$, $t_{s^{-1}} : M \to M, m \to s^{-1}*m$, which satisfy: $t_{s^{-1}}(t_s(m)) = s^{-1}*(s*m) = (s^{-1} \cdot s)*m = e*m = m$, so $t_{s^{-1}} \circ t_s = \mathrm{Id}_M$, and similarly $t_{s^{-1}} \circ t_s = \mathrm{Id}_M$. Therefore t_s is an "unchange" function, so t_s is exactly an element of Sym(M). This examination creates a function

$$\rho : G \to \mathrm{Sym}(M)$$

$$s \mapsto t_s.$$

Now $t_s \cdot s'(m) = (s \cdot s')*m = s*(s'*m)$ by (A2), which is $t_s(t_{s'}(m))$. This holds for all $m \in M$, so $t_s \cdot s' = t_s \circ t_{s'}$, or in other words $\mathfrak{Ɔ}(s \cdot s') = \mathfrak{Ɔ}(s) \circ \mathfrak{Ɔ}(s')$, so $\mathfrak{Ɔ}$ is a homomorphism by definition.

In contrast, this paper constructs a homomorphism $\mathfrak{Ɔ} : G \to \mathrm{Sym}(M)$, and let's define a function

$$* : G \times M \to M$$

$$(s, x) \to \rho(s)(m)$$

Now $e*m = \mathfrak{Ɔ}(e)(m) = \mathrm{Id}(m) = m$, so satisfy identity. Moreover

$$(t \cdot y)*m = \mathfrak{Ɔ}(t \cdot y)(m) = (\mathfrak{Ɔ}(t) \circ \mathfrak{Ɔ}(y))(m) = \mathfrak{Ɔ}(t)(\mathfrak{Ɔ}(y)(m)) = t*(y*m)$$

so associativity is satisfied.

3. Relation

A binary relation ~ defined on a set V is essentially a collection of ordered pairs that are drawn from the Cartesian product $V \times V$.

For any elements a and b in the set V, the pair (a, b) belongs to the relation ~.

A relation ~ defined on a set V is referred to as a reflexive relation when it satisfies the condition that for every element a in the set S, the relation holds true $(a \sim a)$.

A relation denoted by ~ on a set V is considered symmetric if for every pair of elements a and b in S, whenever $a \sim b$ holds true, it follows that $b \sim a$ also holds.

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A relation ~ defined on a set V is considered transitive if, for any elements a, b, and c in the set, whenever a ~ b and b ~ c, it follows that a ~ c.

a, b, c are elements of the set S.

A relation ~ defined on a set V is considered an equivalence relation precisely when it satisfies the properties of reflexivity, symmetry, and transitivity [3].

Given an equivalence relation ~ defined on a set S, for an element a in V, the equivalence class of a is represented as a set.

The set $\{g \in V \mid a \sim g\}$ is represented as $\bar{a}$ or $[a]$.

3.1. Example of equivalence relation

Proposition:

Conjugate relation ~ (a ~ b if $g = k^{-1}bk$ for g, b ∈ G, some k ∈ G) on a group is equivalence relation

Proof.

take $k = e_G$ then a = e · a · e = e ^ {−1} · a · e (reflexive)

$$a = k^{-1}bk$$

$$kak^{-1} = k(k^{-1}bk)k^{-1}$$

$$kak^{-1} = b$$

$$(k^{-1})^{-1}a(k^{-1}) = b \qquad \text{(symmetric)}$$

Let $a = k_1^{-1}bk_1(a\sim b)$, $b = k_2^{-1}ck_2(b \sim c)$

$$\text{a} = \qquad k_1^{-1}bk_1 = k_1^{-1}(k_2^{-1}ck_2)k_1 = (k_1^{-1}k_2^{-1})c(k_2 k_1) = (k_2k_1)^{-1}c(k_2k_1) \qquad \text{(transitive)}$$

3.2. Example of equivalence class

Define a relation ~ on Z, let $n \in Z^+$, for y, j ∈ Z, y ~ j if n|(y − j).

n|(y − y), y~y (reflexive)

If y~j, n|(y − j), then n| − (y − j), n|(j − y), so j~y (symmetric)

If y~j, j~i , for y, j, i ∈ Z. Let $k_1n = (y - j)$, $k_2n = (j - i)$

$k_1n + k_2n = (k_1 + k_2)n = (y - j) + (j - i) = y - j + j - i = (y - i)$, so y ~ i (transitive)

The equivalence class of 0, 1, 2 when n = 3:

$\bar{0} := \{b \in Z : 3 \mid 0 - b\} = 3Z$

$\bar{1} := \{b \in Z : 3 \mid 1 - b\} = 3Z + 1$

$\bar{2} := \{b \in Z : 3 \mid 2 - b\} = 3Z + 2$

This is why elements of Z_3 are $\bar{0}, \bar{1}, \bar{2}$

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4. Orbit, stabilizer, kernel

4.1. Definition of orbit, stabilizer and kernel

Let G act on V.

(i) Orbit of $r \in V$ is Orb(r)

$\mathrm{Orb}(r) := \{y \in V \mid y = g*r,\ g \in G\}$.

The group action is called transitive if $\mathrm{Orb}(r) = V$, for any $r \in V$.

(ii) Stabilizer of $r \in V$ is Stab(r).

$$\mathrm{Stab}(r) := \{y \in G \mid y*r = r\} \leq G$$

Simply, stabilizer is the set of all group elements that fix r.

(iii) Kernel (of the action)

$\ker(q) = \{y \in G \mid y * r = r \text{ for any } r \in V\} = \bigcap_{r \in V} Stab(r)$ [1, 3]

4.2. Theorem

Let G act on L.

(i) For each $l \in L$, $\mathrm{Stab}(l) \leq G$.

(ii) The orbit $\{\mathrm{Orb}(l)\}$ partition L [1, 2].

Part (i), let $a, b \in \mathrm{Stab}(l)$. First notice that

$$l = e*l = \left(b^{-1} \cdot b\right)*l = b^{-1}*\left(b*l\right) = b^{-1}*l \quad ,$$

as $b * l = l$, so $b^{-1} \in Stab(l)$. Now

$$\left(a \cdot b^{-1}\right)*l = a*\left(b^{-1}*l\right) = a*l = l$$

therefore $a \cdot b^{-1} \in Stab(l)$, and hence $\mathrm{Stab}(l) \leq G$

For part (ii)

Partition: Let L be a set and I be an index set, define collection of subsets A_λ, $\lambda \in I$, I is an index set, partition L if:

(*) $A_\lambda \neq \emptyset$ for all $\lambda \in I$.

(**) $U_{\lambda \in I} A_\lambda = L$

(***)if $A_\lambda \bigcap A_\mu \neq \emptyset$, then $\lambda = \mu$

Lemma 4.2.1: Let ~ be an equivalence relation on a set L, then ~ equivalence classes partition L [1, 5].

Proof. $a \in L$, sinc $a = \{s \in L \mid a \sim s\}$, $a \sim a$ by reflexive, therefore $a \in \bar{a}$, $\bar{a} \neq \emptyset$.

$a_\lambda \in \bar{a}_\lambda$ for some $a \in L$, $\lambda \in I$, so every element of L belong to a least one subset, then $U_{\lambda \in I} \bar{a}_\lambda = L$. Suppose $\bar{a}_\lambda \bigcap \bar{a}_\mu \neq \emptyset$, take $x \in \bar{a}_\lambda \bigcap \bar{a}_\mu$, $a_1 \in \bar{a}_\lambda$, $a_2 \in \bar{a}_\mu$. $a_\lambda \sim a_1$, $a_1 \sim a_\lambda$ (symmetric) $a_\lambda \sim x$ so $a_1 \sim x$ (transitive), $a_\mu \sim a_2$, $a_\mu \sim x$, so $a_2 \sim x$ (transitive), $x \sim a_2$ (symmetric)

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then $a_1 \sim a_2$ (transitive), $a_2 \sim a_1$ (symmetric) $a_1 \in \bar{a}_\mu$ for all $a_1 \in \bar{a}_\lambda$, so $\bar{a}_\lambda \subseteq \bar{a}_\mu$. $a_2 \in \bar{a}_\lambda$ for all $a_2 \in \bar{a}_\mu$, so $\bar{a}_\mu \subseteq \bar{a}_\lambda$. $\bar{a}_\mu = \bar{a}_\lambda$, so $\lambda = \mu$. □

Lemma 4.2.2: Let G be a group, L be a set, u ~ r if $g \cdot u = r$, $g \in G$, u, $r \in L$, then u~r is a equivalence relation on L

Proof. $e \cdot u = u$, u ~ u for all $u \in L$ (reflexive).

if u ~ r, there pertain a $g \in G$ such that $g \cdot u = r$, $g^{-1} \cdot g \cdot u = g^{-1} \cdot r = u$, $g^{-1} \in G$, so r~u (symmetric).

Suppose u ~ r, r~o, u, r, $o \in L$ then $g_1 \cdot u = r$, $g_2 \cdot r = o$, for some g_1, $g_2 \in G$, $g_2 \cdot (g_1 \cdot u) = g_2 \cdot r = (g_2 g_1) \cdot u = o, g_2 g_1 \in G$, so u ~ o .(transitive) □

Let G be a group now and L be a set, u ~ r if $g \cdot u = r$, $g \in G$, u, $r \in L$, then u~r is a equivalence relation on L by **Lemma 4.2.2**, equivalence class of a :

$\bar{a} = g \cdot a | g \in G = Orb(a)$, by **Lemma 4.2.1**, $\bar{a}$ partition L, therefore orbits partition L .

5. Lagrange's theorem

Lagrange's Theorem: If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$, and specifically $|G| = [G : H] \cdot |H|$ [1, 5].

Proof. Define a relation ~ on a set G, s ~ w, if and only if $s^{-1}w \in H$.

Reflexive:For any $s \in G$, $s^{-1}s = e$. H is a subgroup, so $e \in H$. So $s \sim s$.

Symmetric: If $s \sim w$, then $s^{-1}w \in H$. Because H is closed under inverses, $(s^{-1}w)^{-1} \in H$, which is $w^{-1}s \in H$. Hence, w ~ s

Transitive: s ~ w, w ~ c, $s^{-1}w \in H$, $w^{-1}c \in H$, *then* $s^{-1}w \cdot w^{-1}c \in H$, $s^{-1}c \in H$. Hence, s ~ c

As ~ is an equivalence relation, it partitions the set G by **Lemma 4.2.1**

The equivalence class of $s \in G$: $[s] = \{z \in G | s \sim z\} = \{z \in G : s^{-1}z \in H$ = sH . So the equivalence classes are precisely the left cosets of H in G . Hence, the equivalence class partitions the set G into equivalence classes and they don't intersect with each other(Lemma 4.2.1), H partition G

Take any left coset $sH = \{sh : d \in H\}$. Define a function $o : H \to sH$, $f(d) = sd$

take $d_1, d_2 \in H$, $f(d_1) = sd_1 = f(d_2) = sd_2, s^{-1}sd_1 = s^{-1}sd_2 = d_1 = d_2$. (injective)

By definition, every element of sH can be express as ah for some $d \in H$. So for any element in sH, there is a preimage $d \in H$. (surjective)

since o is bijective, $|sH| = |H|$ [3]

This paper established that the left cosets partition G. Suppose there are exactly k distinct left cosets. (This number k is, by definition, the index $[G : H]$). Since every coset has exactly $|H|$ elements, and these cosets are pairwise disjoint, the overall count of elements in G is: $|G| = k \cdot |H| = [G : H] \cdot |H|$

Its main strength lies in restricting potential subgroup structures without complex calculations. Beyond pure algebra, it provides the essential partitioning framework that enables key results like the Orbit-Stabilizer Theorem and Burnside's Lemma.

6. Orbit-stabilizer theorem

Orbit-Stabilizer Theorem: Let G be a finite group acting on a set K. For any $ķ \in K$: $|G| = |\mathrm{Orb}(ķ)| \cdot |\mathrm{Stab}(ķ)|$

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Proof.Equivalently, $|\mathrm{Orb}(ķ)| = [\mathrm{G} : \mathrm{Stab}(ķ)]$, the index of the stabilizer in G. Knowing from **4.2 (ii)** that the orbits partition K . Also knowing from Theorem 4.2(i) that $\mathrm{Stab}(ķ) \leq \mathrm{G}$. Since G is finite, **Lagrange's Theorem** applies, so:

$|\mathrm{G}| = |\mathrm{G} : \mathrm{Stab}(ķ)| \cdot |\mathrm{Stab}(ķ)|$. Therefore it is sufficient to prove that

$$[\mathrm{G} : \mathrm{Stab}(ķ)] = |\mathrm{Orb}(ķ)|$$

First, let $\mathrm{B} = \mathrm{Stab}(ķ)$. Consider the set of left cosets of B in G : $\mathrm{G}/\mathrm{B} = \{\mathrm{gB} : \mathrm{g} \in \mathrm{G}\}$

define a function $\varphi : \mathrm{G}/\mathrm{B} \to \mathrm{Orb}(ķ)$ by $\varphi(\mathrm{gb}) = \mathrm{g} \cdot ķ$. Suppose $\mathrm{gB} = \mathrm{g'B}$. As $\mathrm{g'} \in \mathrm{gB}$, there persists $\mathrm{b} \in \mathrm{B}$ such that $\mathrm{g'} = \mathrm{gb}$. Since $\mathrm{b} \in \mathrm{Stab}(ķ)$, $\mathrm{b} \cdot ķ = ķ$.

$$\varphi(\mathrm{g'B}) = \mathrm{g'} \cdot ķ = (\mathrm{gb}) \cdot ķ = \mathrm{g} \cdot (\mathrm{b} \cdot ķ) = \mathrm{g} \cdot ķ = \varphi(\mathrm{gB})$$

Thus, φ gives the same output regardless of which representative this study chooses for the coset. So φ is well-defined.

Suppose $\varphi(\mathrm{gB}) = \varphi(\mathrm{g'B}) = \mathrm{g} \cdot ķ = \mathrm{g'} \cdot ķ$, $g^{-1}g' \cdot ķ = g^{-1}g \cdot ķ = e \cdot ķ = ķ$, *this means* $g^{-1}g' \in Stab(ķ) = B$, then $g^{-1}g'B = B$, $gg^{-1}g'B = gB = g'B$. (injective)

Take any element $\mathrm{y} \in \mathrm{Orb}(ķ)$. By the definition of an orbit, there exists some $\mathrm{g} \in \mathrm{G}$ such that $\mathrm{y} = \mathrm{g} \cdot ķ$. But then: $\mathrm{y} = \mathrm{g} \cdot ķ = \varphi(\mathrm{gB})$. (surjective)[1, 3]

Since φ is a bijection, the quantity of left cosets of $\mathrm{Stab}(ķ)$ in G equals the number of elements in the orbit $\mathrm{Orb}(ķ)$: $[\mathrm{G} : \mathrm{Stab}(ķ)] = |\mathrm{Orb}(ķ)|$

Now applying **Lagrange's Theorem:** $(|\mathrm{G}| = [\mathrm{G} : \mathrm{Stab}(ķ)] \cdot |\mathrm{Stab}(ķ)|)$

$|\mathrm{G}| = |\mathrm{Orb}(ķ)| \cdot |\mathrm{Stab}(ķ)|$ [1, 2]

The Orbit-Stabilizer Theorem serves as a cornerstone of group theory by connecting abstract algebraic operations with the geometric structure of symmetry. By establishing that the order of a group acting on a set equals the product of the size of an element's orbit and its stabilizer subgroup, the theorem provides a precise formula to quantify symmetry without double-counting due to symmetrical transformation. Its primary impact lies in transforming complex combinatorial counting problems into manageable calculations, acting as the mathematical foundation for key results like Burnside's Lemma and Pólya's Enumeration Theorem. Furthermore, it extends beyond theoretical mathematics into computational group; for instance, it allows algorithms like the Schreier-Sims algorithm to build stabilizer chains, allowing computers to compute exact reachable state spaces and solve complex permutation puzzles like the Rubik's Cube

7. Burnside Lemma

Burnside Lemma: Let G be a finite group acting on a set M. Defining $\mathrm{Fix(g)} = \{м \in \mathrm{M} : \mathrm{g} * м = м\}$ as elements of M fixed by element g in G, then the number of distinct orbits of the action is $\frac{1}{|G|} \sum_{g \in \mathrm{G}} |Fix(g)|$ [2, 6].

Proof. Let $\mathrm{S} = \{(\mathrm{g}, м) \in \mathrm{G} \times \mathrm{M} : \mathrm{g} * м = м\}$, then $\left|\mathrm{S}\right| = \sum_{g \in \mathrm{G}} |Fix(g)|$, This paper observes that $\sum_{м \in \mathrm{M}} |\mathrm{Stab}(м)| = |\mathrm{S}|$, so therefore have $\sum_{м \in \mathrm{M}} |\mathrm{Stab}(м)| = \sum_{g \in \mathrm{G}} |Fix(g)|$. By Orbit-Stabilizer Theorem, $|\mathrm{G}| = |\mathrm{Orb}(м)| \cdot |\mathrm{Stab}(м)|$, then $\left|\mathrm{Stab}\left(м\right)\right| = \frac{|G|}{|Orb(м)|}$, substitute

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this in previous equation, $\sum_{g\in G}|Fix(g)|=\sum_{м\in M}\frac{|G|}{|Orb(м)|}$. This paper defines $|Orb(м)|=|R|$, for every $м$ from orbit a R, $\left|Stab\left(м\right)\right|=\frac{|G|}{|R|}$, since $\frac{|G|}{|R|}$ is a constant number, $\sum_{м\in R}|Stab(м)|=\frac{|G|}{|R|}\cdot\sum_{м\in R}1=\frac{|G|}{|R|}\cdot\left|R\right|=\left|G\right|$. From Lemma 4.2.1, orbit partition M , let N be the number of orbits so $\sum_{м\in M}\frac{|G|}{|Orb(м)|}=\sum_{orbits\ R}\sum_{м\in R}\frac{|G|}{|R|}=\sum_{orbits\ R}|G|=N\cdot|G|=\sum_{м\in M}|Stab(м)|$. Ultimately, $\frac{1}{|G|}\sum_{м\in M}|Stab(м)|=N$

This Lemma allows to count configurations of an object while treating situations that can be transformed to another via group action. Especially when facing a problem of enumerating symmetric objects. Moreover, it builds up a connection between local group properties and global geometry. Burnside's Lemma simplifies the process of counting the configurations by partitioning the set in a trivial way, which involves using the stabilizer in group action.

Problem: How may rotationally distinct colourings of the face of a cube three colours?

Let X be the set of 3^6 possible colouring combinations of a fix cube, Let G be the group that applies on the cube which rotate the cube: Tow colorings in X pertain to the same orbit if one is a rotation of another. There are 24 elements in G: one identity element fixed all 3^6 situation, six 90° rotations about axis through opposite faces each fix 3^3 colourings, three 180° rotations about axis through opposite faces each fix 3^4 configurations, eight 120° rotations about axis through opposite vertices each fix 3^2 conditions, six 180° rotations about opposite edges each fix 3^3 [3, 6]

Number of distinct states = $\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$

The number of rotationally same states is thus:

$$\frac{1}{24}\left(3^6+6\times3^3+3\times3^4+8\times3^2+6\times3^3\right)=57$$

As a result, there are 57 rotationally distinct colourings of the faces of the cube with three colours.

This application shows a basic process of how Burnside's Lemma is used in a simple geometrical puzzle [7-10]. The serviceability of Burnside's Lemma can be transferred to other intricate counting problems.

8. Conclusion

This paper built the foundational framework of group actions, from the basic rules to fundamental results, further into Lagrange's Theorem, thereby deriving Burnside's Lemma. Finally, Burnside's Lemma was applied to a classic combinatorial enumeration problem. The theoretical frameworks established in this paper provide a rigorous foundation for advanced results, such as the Pólya Enumeration Theorem, extending to linear group actions and representation theory, or computational group theory, like the Schreier-Sims Algorithm. Through the study of group actions, the relationship between symmetry and counting problems becomes clearer, showing how abstract algebraic concepts can be used to solve practical mathematical problems. The Orbit-Stabilizer Theorem connects the structure of a group with the behavior of elements under group actions, while Burnside's Lemma further provides an effective method for counting different configurations under

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symmetry. These results not only demonstrate the power of group theory in combinatorics but also show the importance of equivalence relations and partitions in mathematical analysis. Although this paper mainly focuses on basic theories and a simple application, the methods introduced can be extended to more complex problems in algebra, geometry, and computer science. In particular, symmetry analysis and group-based methods continue to play an important role in areas such as puzzle solving, algorithm design, and computational mathematics. Therefore, the concepts discussed in this paper serve as an important starting point for further study of advanced topics in group theory and its applications.

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