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Mathematical Structures in Musical Rhythm, Tuning Systems, and Sound Waves

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Abstract. Music is generally considered an artistic form of expression, but many of its fundamental structures can also be described mathematically. Rhythm depends on numerical patterns and divisions of time, musical tuning is based on relationships between frequencies, and sound itself can be modeled using periodic functions. This paper uses a literature review method to examine the applications of mathematics in three major areas of music: rhythm, tuning systems, and sound waves. Mathematical concepts including ratios, sequences, exponential and logarithmic functions, trigonometric functions, and Fourier analysis are discussed together with their musical applications. The review shows that mathematical models successfully explain many structural and physical properties of music. In particular, discrete patterns can represent rhythm, exponential relationships explain equal temperament, and combinations of sinusoidal waves describe musical timbre. However, mathematical models alone cannot fully explain musical perception, cultural preference, or aesthetic experience.

Keywords: mathematics, musical rhythm, tuning systems, sound waves

1. Introduction

Mathematics and music have been connected for centuries. Although music is usually experienced through hearing and emotion, many elements of music have clear numerical structures. Musical notes can be described by frequencies, intervals can be represented through ratios, rhythms can be divided into numerical patterns, and complex sounds can be analyzed as combinations of simpler waves. Harkleroad explains that mathematical ideas ranging from ratios and numerical patterns to Fourier analysis can be used to investigate musical structures [1]. Benson similarly connects mathematical concepts with waves, harmonics, scales, temperament, and musical symmetry [2].

These relationships show that mathematics is not simply an external tool applied to music. Instead, mathematics provides a language for describing some of the structures that make music possible. For example, rhythm depends on the division of time, pitch depends on the frequency of vibration, and musical timbre depends on combinations of different frequency components.

This paper explores the application of mathematics in three key aspects of music: rhythm, tuning systems, and sound waves. Utilizing a literature review, the research examines areas such as music theory and acoustics, focusing on concepts like ratios, sequences, algorithms, and Fourier analysis.

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These insights aim to enhance music education, acoustics, instrument design, and interdisciplinary mathematics education.

2. Mathematics in musical rhythm

2.1. Numerical representation of rhythm

Rhythm is one of the clearest examples of mathematical organization in music because it depends on the division of time. In traditional musical notation, note lengths are based on proportional relationships. If a whole note is assigned a relative duration of 1, then a half note has a duration of 1/2, a quarter note has a duration of 1/4, and an eighth note has a duration of 1/8. These proportions allow musicians to organize sound into structured units.

Tempo can also be described mathematically. Beats per minute, or BPM, indicates the number of beats that occur during sixty seconds. If the tempo is B beats per minute, the duration T of one beat can be calculated as $T = 60 / B$, where T is measured in seconds. For example, if a piece has a tempo of 120 BPM, $T = 60 / 120 = 0.5$ seconds. Therefore, every beat lasts approximately half a second. If the BPM increases, the duration of each beat decreases. This is a simple example of an inverse mathematical relationship appearing directly in musical performance.

Musical meter also depends on repeated time structures. A 4/4 measure contains four quarter-note beats, while a 3/4 measure contains three quarter-note beats. However, rhythm involves more than simply counting beats. Listeners also organize sounds into repeating metric patterns. London describes musical meter as involving synchronization between attention, movement, and repeated temporal structures [3]. Therefore, mathematics can explain the timing of rhythm, while perception helps listeners organize that timing into meaningful patterns.

2.2. Binary patterns and euclidean rhythms

Rhythms can also be represented using ideas from discrete mathematics. A sequence of equally spaced musical pulses can be described using binary values. A musical onset can be represented by 1, while a rest can be represented by 0. For example, $1 =$ musical onset ,, $0 =$ rest . An eight-pulse rhythm may therefore be represented as $[1, 0, 0, 1, 0, 0, 1, 0]$. Instead of writing the rhythm in conventional musical notation, the same information is expressed as a mathematical sequence.

An especially interesting relationship between mathematics and rhythm appears in Euclidean rhythms. Toussaint demonstrated that the structure of the Euclidean algorithm, which is normally used to calculate the greatest common divisor of two integers, can also generate rhythmic patterns found in different musical traditions [4]. A Euclidean rhythm may be written as $E(k, n)$, where k represents the number of musical onsets and n represents the total number of pulses. For example,

$$E(3, 8) = [1, 0, 0, 1, 0, 0, 1, 0] \tag{1}$$

Here, three musical onsets are distributed among eight possible positions as evenly as possible. This creates a pattern related to the common 3-3-2 rhythmic organization found in several musical traditions.

What makes this example significant is that the Euclidean algorithm was not originally created for music. It was developed as a mathematical procedure for number theory. However, the same mathematical principle can create patterns that appear naturally in musical rhythm. This demonstrates how mathematical structures can appear in areas that initially seem unrelated.

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2.3. Mathematics in larger musical structures

Mathematical organization can also appear at larger levels of a musical composition. Repetition, symmetry, grouping, and periodicity help listeners recognize musical sections and develop expectations.

For example, a four-measure phrase may repeat several times. In this case, the musical structure is periodic because a pattern returns after a fixed amount of time. A rhythm may also contain smaller repeated patterns inside larger repeated patterns. Lerdahl et al. developed a theory of tonal music in which listeners understand musical structure through different hierarchical levels, including rhythmic grouping and meter [5]. This suggests that rhythm can be modeled as a system of nested patterns.

Therefore, mathematics can describe rhythm at several levels. Fractions and ratios represent note durations, binary sequences represent patterns of sounds and rests, algorithms can generate rhythmic structures, and hierarchical patterns can describe larger musical forms.

3. Mathematics in musical tuning

3.1. Frequency ratios and musical intervals

Pitch is closely related to the frequency of a sound wave. Frequency describes the number of vibrations or cycles that occur every second and is measured in hertz, or Hz. Generally, a higher frequency produces a higher perceived pitch.

The relationships among frequencies form the mathematical basis of tuning systems. Some important musical intervals can be represented using simple numerical ratios. For example, an octave has a frequency ratio of $2:1$. If one note has a frequency of 220 Hz, the note one octave above it has a frequency of approximately 440 Hz. A perfect fifth can be represented by the ratio $3:2$, while a perfect fourth can be represented by the ratio $4:3$.

The relationship between simple numerical proportions and musical intervals has played an important role in music theory for centuries [6]. Systems such as Pythagorean tuning and just intonation use simple frequency ratios to construct musical scales. However, a problem occurs when musicians attempt to use simple ratios across many different keys. It is mathematically impossible to make every interval in every key perfectly match simple frequency ratios. This creates a conflict between mathematical purity and musical flexibility.

3.2. Twelve-tone equal temperament

One important solution is twelve-tone equal temperament, which is widely used in modern Western music. In this system, one octave is divided into twelve equal semitone steps. Suppose the starting frequency is f_0 and the frequency one octave above it is f_{12}. Since an octave doubles the frequency, $f_{12} = 2f_0$. If each semitone changes the frequency by the same ratio r, then after twelve semitones, $r^{12} = 2$. Therefore, $r = 2^{1/12}$.

The frequency of a note n semitones above the starting note can then be calculated using

$$f_n = f_0 \times 2^{n/12} \tag{2}$$

This formula shows that musical pitch in equal temperament follows an exponential relationship. For example, if A4 has a frequency of 440 Hz, then the A one octave above it has a frequency of

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$$f_{12} = 440 \times 2^{12/12} \tag{3}$$

$$f_{12} = 440 \times 2 \tag{4}$$

$$f_{12} = 880\ \text{Hz} \tag{5}$$

The frequency one semitone above 440 Hz can be calculated as

$$f_1 = 440 \times 2^{1/12} \tag{6}$$

$$f_1 \approx 466.16\ \text{Hz} \tag{7}$$

This demonstrates that equal musical steps do not correspond to adding the same number of hertz. Instead, each frequency is multiplied by the same constant ratio. Equal temperament makes it possible for musicians to play in many different keys without completely retuning an instrument. This is especially important for instruments such as the piano, where the pitches are fixed during performance. Sethares argues that tuning should also be considered together with timbre and sound spectrum [7]. The perception of consonance may depend not only on pitch ratios but also on the harmonic structure of the sound being produced.

3.3. Logarithms and musical intervals

Since pitch relationships depend on ratios, logarithms are useful for measuring the size of musical intervals. The cent is one widely used unit. One octave contains 1200 cents, while one equally tempered semitone contains 100 cents. The difference in cents between two frequencies f_1 and f_2 can be calculated using

$$C = 1200 \times \log_2\left(\frac{f_2}{f_1}\right) \tag{8}$$

For example, suppose f_2 is exactly twice f_1. Then, $\frac{f_2}{f_1} = 2$. Therefore,

$$\frac{f_2}{f_1} = 2.)C = 1200 \times log_2(2) \tag{9}$$

$$C = 1200 \times 1 \tag{10}$$

$$C = 1200\ \text{cents} \tag{11}$$

This confirms that a frequency ratio of $2:1$ represents one octave. Exponentials and logarithms therefore play complementary roles in musical tuning. Exponential functions can be used to generate frequencies, while logarithmic functions can be used to measure the distance between pitches. Loy discusses frequency, pitch, intervals, scales, and other mathematical properties of musical sound as part of the mathematical foundations of music [8].

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4. Mathematics in sound waves

4.1. Trigonometric functions and simple sound waves

Music physically exists as vibration. When a guitar string, piano string, vocal cord, or column of air vibrates, it creates changes in air pressure. These pressure variations travel through the air as sound waves. A simple periodic sound wave can be modeled using a sine function:

$$y(t) = A\sin(2\pi ft + \varphi) \tag{12}$$

In this equation, A = amplitude, f = frequency, t = time, φ = phase. Frequency indicates how many complete cycles occur every second. The period T represents how long one complete cycle takes.

The relationship between period and frequency is $T = 1/f$. For example, if a musical note has a frequency of 440 Hz, T = 1 / 440, T ≈ 0.00227 seconds. This means that one complete cycle of the wave lasts approximately 0.00227 seconds.

Amplitude is connected to the magnitude of the vibration, while frequency is strongly connected to pitch. Although real musical sounds are usually much more complicated than a single sine wave, sinusoidal functions provide a useful mathematical model for understanding basic acoustics. Fletcher et al. use mathematical and physical principles to analyze sound waves, strings, guitars, pianos, and many other musical instruments [9].

4.2. Harmonics and superposition

Real musical instruments do not normally produce only one frequency. Instead, they produce a fundamental frequency together with additional frequencies called harmonics. If the fundamental frequency is f_0, the harmonic frequencies can be approximately represented as $f_0,\ 2f_0,\ 3f_0,\ 4f_0,\ \ldots$

The fundamental frequency usually has the strongest relationship to the pitch that a listener hears. The higher harmonics help determine the timbre, or tone quality, of the instrument. This explains why two different instruments can play the same musical note but still sound different. A guitar and a piano may both play a note with a fundamental frequency of 440 Hz, but the relative strengths of their higher harmonics may be different.

The principle of superposition allows multiple waves to be added together mathematically. A simplified sound containing three harmonic components may be represented as

$$x(t) = A_1\sin(2\pi f_0 t) + A_2\sin(2\pi(2f_0)t) + A_3\sin(2\pi(3f_0)t) \tag{13}$$

Here, A_1, A_2, and A_3 represent the amplitudes of the different frequency components.

If these amplitudes change, the shape of the final waveform also changes. Therefore, even when two sounds have the same fundamental frequency, different harmonic combinations can create different timbres. Hall explains that musical acoustics connects the physical behavior of sound with musical perception and the behavior of instruments [10].

4.3. Fourier analysis

The idea of combining several sine waves leads to Fourier analysis. Fourier analysis shows that a complicated periodic waveform can be represented as a combination of simpler sine and cosine waves.

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A general Fourier series can be written as

$$x(t) = a_0 + \sum [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \tag{14}$$

$$\text{for } n = 1,2,3,\ldots \tag{15}$$

In this expression, a_0 is a constant term, while a_n and b_n describe the strengths of different frequency components. The value ω_0 represents the fundamental angular frequency. The major idea is that a complex sound can be separated into individual frequencies. This is especially useful in modern digital audio.

For example, a recorded guitar note may look very complicated when represented as a waveform over time. However, frequency analysis may show several clear peaks. The first large peak may represent the fundamental frequency, while additional peaks may represent higher harmonics. Fourier analysis therefore allows researchers and audio engineers to examine the internal frequency structure of sound.

It has applications in areas such as audio equalization, sound synthesis, instrument recognition, recording technology, and digital signal processing. Benson identifies Fourier theory as one of the major mathematical connections between music and acoustics [2].

4.4. Mathematical modeling of string instruments

String instruments provide another clear example of mathematics in music. The frequency of an ideal vibrating string can be approximated using

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{16}$$

In this equation, f_n = frequency of the nth harmonic, n = harmonic number, L = vibrating length of the string, T = tension of the string, μ = linear mass density of the string. This formula explains several familiar characteristics of guitar playing.

First, frequency is inversely related to string length. When a guitarist presses a fret, the vibrating length of the string becomes shorter. Therefore, the frequency increases and the pitch becomes higher. Second, frequency increases when string tension increases. Turning a tuning peg tightens the string and raises its pitch.Third, a string with greater mass per unit length generally produces a lower frequency if the other variables remain unchanged. This is one reason why thicker guitar strings usually produce lower pitches.

The formula also shows that the relationship between tension and frequency is not linear. Frequency is proportional to the square root of tension. For example, if the tension becomes four times greater, $\sqrt{4} = 2$, so the frequency approximately doubles. These relationships demonstrate how algebra and physical modeling can explain the behavior of actual musical instruments.

5. Limitations and future development of mathematics and music

Although mathematics can describe many important properties of music, it cannot completely explain the musical experience. Mathematical models usually simplify real musical systems.

For example,

$$y(t) = A\sin(2\pi ft + \varphi) \tag{17}$$

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represents an ideal periodic sound wave. A real guitar note is more complicated. Its amplitude changes over time, the note has an attack and decay, and the sound may contain noise and other nonlinear effects.

Similarly, simple frequency ratios cannot completely determine whether a listener finds an interval pleasant. Musical perception is influenced by culture, musical training, historical traditions, timbre, and individual experience. Sethares shows that consonance is affected by the relationship among tuning, timbre, and spectral structure [7].

Mathematical models also have difficulty describing the emotional meaning of music. Two pieces may use similar rhythms, frequencies, and harmonic structures while creating very different emotional responses. Mathematics is therefore most useful for describing measurable structures rather than completely explaining aesthetic experience.

Future research can further connect mathematics and music through digital signal processing, computational music analysis, algorithmic composition, and sound synthesis [11]. Researchers could also combine mathematical theory with experimental data. For example, notes from different guitars could be recorded and analyzed using Fourier techniques. Researchers could compare their fundamental frequencies, harmonic spectra, and amplitude decay. Such studies could provide a stronger connection between mathematical theory and actual musical performance.

6. Conclusion

This paper reviews the applications of mathematics in musical rhythm, tuning systems, and sound waves. The analysis shows that different mathematical concepts can explain different parts of music. In rhythm, fractions and ratios describe note durations, binary sequences represent rhythmic patterns, and algorithms such as the Euclidean algorithm can generate musical structures. In musical tuning, frequency ratios describe intervals, while exponential and logarithmic functions explain equal temperament and the measurement of pitch differences. In musical acoustics, trigonometric functions model periodic vibrations, harmonic frequencies help explain timbre, and Fourier analysis separates complex sounds into simpler frequency components.

Overall, mathematics provides a powerful language for describing the structural and physical foundations of music. It allows musical patterns and acoustic phenomena to be represented quantitatively and compared systematically. However, mathematical models cannot completely explain how music is experienced because perception is also influenced by cultural, psychological, and aesthetic factors.

A limitation of this paper is that it mainly uses a literature review and lacks original experimental data. Future research could analyze recordings from real musical instruments and compare their frequencies, harmonics, and amplitude changes. This could further connect mathematical theory with musical performance, acoustics, and digital audio technology.

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